



Minima and best approximations in constructive analysis

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Abstract: Working in Bishop’s constructive mathematics, we first show that minima can be defined as best approximations, in such a way as to preserve the compactness of the underlying metric space when the function is uniformly continuous. Results about finding minima can therefore be carried over to the setting of finding best approximations. In particular, the implication from having at most one best approximation to having uniformly at most one best approximation is equivalent to Brouwer’s fan theorem for decidable bars. We then show that for the particular case of finite-dimensional subspaces of normed spaces, these two notions do coincide. This gives us a better understanding of Bridges’ proof that finite-dimensional subspaces with at most one best approximation do in fact have one. As a complement we briefly review how the case of best approximations to a convex subset of a uniformly convex normed space fits into the unique existence paradigm.

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1 Introduction

The present paper is set in the realm of Bishop’s constructive mathematics (**BISH**)¹. Bishop’s constructive framework is distinguished from classical—that is traditional—mathematics by a strict adherence to intuitionistic logic. This allows us to view **BISH** as a generalisation of classical mathematics [43]: adding the law of excluded middle recovers classical logic.

To start with we recall some background material. Let (X, d) be a metric space. Any such X is *compact* if it is totally bounded and complete, where being *complete* means

¹We refer to [13, 14, 21, 24] for a development of constructive analysis in **BISH**, and to [1, 16, 27, 42] for constructive alternatives to ZFC.

that every Cauchy sequence converges. Moreover, a subset S of X is *located* if for each $z \in X$ the distance

$$\text{dist}(z, S) = \inf_{x \in S} d(x, z)$$

from x to S exists. Any totally bounded subset of a metric space is located.

Continuity is a thorny issue in constructive mathematics². Since the three major models of **BISH**—classical mathematics, Brouwer’s intuitionistic mathematics, and the Russian school of recursive mathematics³—all disagree on which continuity principles hold, we must be very careful; in particular, in recursive mathematics there is a continuous function on the unit interval which is not uniformly continuous. In [13], Bishop elegantly sidesteps this problem by including the uniform continuity theorem in the definition of continuity: f is a *Bishop-continuous* function if S is locally totally bounded and f is uniformly continuous on each compact subset of S . Although this solution allowed Bishop and his followers to develop a significant body of results, we will only ever need either pointwise or uniform continuity in this paper; when we refer to a function as continuous we mean that it is pointwise continuous.

In the following we loosely follow [50]. Let f be a continuous real-valued function on (a subset S of) X . For the sake of a simpler presentation we assume that $f \geq 0$. The conditional existence problem

If the infimum of f on S is 0, can we construct $\xi \in S$ such that $f(\xi) = 0$?

incorporates, with $f : S \rightarrow \mathbf{R}$ defined by

$$f(x) = d(x, z) - \text{dist}(z, S),$$

the search for best approximations to a point z of X in a located subset S of X . That is to say, a best approximation to z in S is nothing but a point of S at which the uniformly continuous function $d(\cdot, z) : S \rightarrow \mathbf{R}$ attains its infimum $\text{dist}(z, S)$. This, however, does not mean that a best approximation can always be constructed, even if S is compact—see below.

A natural first step towards solving the general problem is to construct—using countable choice—a sequence (x_n) in S such that

$$f(x_n) < 1/n$$

²See, for instance, [48] for more on continuity in constructive analysis.

³See [21] for an introduction to the intuitionistic and recursive schools of mathematics and their relationship to classical mathematics and Bishop’s constructive mathematics.

for each $n \geq 1$, which is indeed possible whenever f restricted to S has infimum 0. If S is compact, working classically we could then, using the Bolzano-Weierstrass theorem

BWT *Every sequence in a compact metric space has a convergent subsequence,*

select a convergent subsequence of (x_n) , whose limit ξ would moreover satisfy $f(\xi) = 0$. In fact, the minimum theorem

MIN *Every uniformly continuous function on a compact metric space has a minimum,*

which would solve our problem with a single stroke of the pen, is a simple consequence of **BWT**. However, **MIN**, and therefore **BWT**, is essentially non-constructive [28].⁴

The non-constructive character of **MIN** notwithstanding, one can show constructively that

$$\inf f = \inf_{x \in X} f(x)$$

exists whenever $f : X \rightarrow \mathbb{R}$ is uniformly continuous, and X is totally bounded [21, Chapter 2, Theorem 4.5]. So if we restrict ourselves to continuous functions on compact spaces our problem becomes

When does a uniformly continuous function on a compact metric space attain its infimum?

We now drop the assumption that $f \geq 0$, but suppose that $\inf f$ can be computed. A function $f : X \rightarrow \mathbb{R}$ is said to have *uniformly at most one minimum* [49] if

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x, y \in X (\max \{f(x), f(y)\} < \inf f + \delta \Rightarrow d(x, y) < \varepsilon)$$

or equivalently

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x, y \in X (d(x, y) \geq \varepsilon \Rightarrow f(x) \geq \inf f + \delta \vee f(y) \geq \inf f + \delta).$$

If f has uniformly at most one minimum, then f has *at most one minimum* [9]:

$$\forall \varepsilon > 0 \forall x, y \in X \exists \delta > 0 (d(x, y) \geq \varepsilon \Rightarrow f(x) \geq \inf f + \delta \vee f(y) \geq \inf f + \delta)$$

or equivalently

$$\forall x, y \in X (x \neq y \Rightarrow f(x) > \inf f \vee f(y) > \inf f).$$

If f has at most one minimum, then any point—if one exists—at which f attains its infimum is unique; that is, if $x, y \in X$ are such that $f(x)$ and $f(y)$ are both equal to $\inf f$, then $x = y$.

⁴An outline of this, including references, can be found e.g. in [49].

Now let X be complete. As recalled in [49], if f has uniformly at most one minimum, then f actually has a minimum: that is, f attains its infimum.⁵ To prove this, one can construct, with countable choice as in the classical argument discussed before, a sequence (x_n) in X such that

$$f(x_n) < 1/n$$

for each $n \geq 1$. Now the additional hypothesis can be used that f has uniformly at most one minimum, to show that (x_n) is a Cauchy sequence. Since X is complete, (x_n) has a limit ξ in X , for which clearly $f(\xi) = \inf f$.

Hence if X is complete and f has uniformly at most one minimum, then one can construct the *a fortiori* unique minimum of f . Even if X fails to be complete, the given data can be converted into an element of the completion of X : namely, into the Cauchy sequence (x_n) in X . This observation has suggested a way [50]⁶ to get by in Bishop-style constructive mathematics without countable choice as proposed by Richman [44, 45], where completions are defined without sequences.

Being essentially folklore, this problem and its solution have some history. The first known occurrence is in Theorem 4 of Lifschitz's [38]—however, there is no proof in the English translation of the Russian original from 1971. Kreinovich [37] also refers to a related metatheorem by M.G. Gelfond from 1972. The work of Bridges in the 1980s was centred around best approximations; see [19] for an overview and for references. To our knowledge, the next occurrence of this argument is Problem 10, ascribed to Aczel, in Chapter 2 of Bridges and Richman's [21]. Uniform uniqueness in the form of moduli of uniqueness was then used intensively by Kohlenbach from the early 1990s [34], first also in the context of best approximations.⁷

Recently unique existence has further occurred in work by Brattka [15] in computable analysis à la Weihrauch [56], and by Taylor [53] within his abstract Stone duality, see also [2]. In both cases non-uniform uniqueness suffices because versions of the Heine-Borel theorem are available. As noticed in [49], the statement

UAM *If a uniformly continuous function on a compact metric space has at most one minimum, then it has uniformly at most one minimum*

is indeed equivalent⁸ to Brouwer's fan theorem for decidable bars, the paradigmatic version of the Heine-Borel theorem [21], which can be put as⁹

⁵This is analogous to the rule from [34, Theorem 4.4].

⁶Diener and the second author [26] have applied this to the implicit functions theorem.

⁷We refer to Kohlenbach and Oliva's survey [35] and to Kohlenbach's book [36] for more recent developments.

⁸One half of this equivalence corresponds to the rule from [34, Theorem 4.3].

⁹See, for instance, [11, 32]. FT is the classical contrapositive of the Weak König Lemma.

FT *Every decidable binary tree without an infinite path is finite.*

This classification in the spirit of constructive reverse mathematics—as practised by Ishihara [29, 30, 31] and others¹⁰—sharpenes an earlier result the second author has obtained jointly with Berger and Bridges [9]; see also [10, 32, 33, 52].

In the first part of the present note the classification of **UAM** is carried over from the context of minima to the more specific context of best approximations that was considered much earlier. In the second part we have a closer look at a prime case of the latter where a best approximation of a point in a normed space is sought in a finite-dimensional subspace. We conclude with a brief discussion of best approximations in a convex subset to a point in a uniformly convex normed space.

2 Minima and best approximations

We first show how minima can be expressed as best approximations. Let X be a metric space with metric d , let S be a located subset of X , and let x be a point of X . A *best approximation* to x in S is a minimum point of the function $f : S \rightarrow \mathbf{R}$ defined by

$$f(s) = d(x, s).$$

Moreover, x has (uniformly) at most one best approximation in S if f has (uniformly) at most one minimum in S . The following theorem shows conversely that minima of a function on a metric space into \mathbf{R} which is bounded below can be seen as best approximations.

Theorem 1 *Let f be a function from a metric space (X, d) into \mathbf{R} which is bounded below. By $*$ we denote a point that is not in X . There exists a metric ρ on $X \cup \{*\}$ such that*

$$f(x) = \rho(*, x)$$

for every $x \in X$; in particular, the minima of f in X are exactly the best approximations to $$ in X . Moreover, $\inf f$ can be computed if and only if $\text{dist}_\rho(*, X)$ can be computed, in which case*

$$\inf f = \text{dist}_\rho(*, X).$$

If, in addition, (X, d) is bounded, then f has (uniformly) at most one minimum in (X, d) precisely when $$ has (uniformly) at most one best approximation in (X, ρ) . Finally, if (X, d) is compact and f is uniformly continuous, then (X, ρ) is compact.*

¹⁰See, for instance, [3, 4, 5, 6, 7, 8, 11, 12, 20, 25, 39, 40, 41, 54, 55]

Proof Let f be a function from (X, d) into \mathbf{R} which is bounded below, let $*$ be some point not in X , and let $X' = X \cup \{*\}$. We first replace d by the metric $d' = d/(1 + d)$ on X , with respect to which X is bounded with diameter no greater than 1, and define $g : X \rightarrow \mathbf{R}$ by

$$g(x) = f(x) - 1/2;$$

since f is bounded below, we may assume that $g(x) \geq 0$ for all x . We next extend d' to be a metric on X' by setting $d'(*, x) = 1/2$ for all $x \in X$; the only nontrivial property to verify is the following instance of the triangle inequality: for all $x, y \in X$ we have

$$d'(x, y) \leq 1 = d'(x, *) + d'(*, y),$$

because the diameter of (X, d') is bounded by 1. We finally extend g to X' by setting $g(*) = 0$, and define $\rho : X' \times X' \rightarrow \mathbf{R}$ by

$$\rho(x, y) = d'(x, y) + |g(x) - g(y)|,$$

for which clearly $d' \leq \rho$. It is easy to see that $(x, y) \mapsto |g(x) - g(y)|$ is a pseudo-metric on X' ; whence (X', ρ) is a metric space. Moreover,

$$\begin{aligned} \rho(*, x) &= d'(*, x) + |g(x) - g(*)| \\ &= 1/2 + |g(x)| \\ &= 1/2 + g(x) \\ &= f(x) \end{aligned}$$

for every x . In particular, each minimum of f corresponds to a best approximation to $*$ in the subset X of (X', ρ) , and $\text{dist}_\rho(*, X)$ can be computed precisely when $\inf f$ can be computed, in which case they are equal.

We now assume that (X, d) is bounded, and show that f has (uniformly) at most one minimum in (X, d) if and only if $*$ has (uniformly) at most one best approximation in (X, ρ) . Since X is bounded, d and d' are similar metrics (i.e. each of them is bounded by the other up to a nonnegative multiplicative constant), from which—together with $d' \leq \rho$ —the “if” part of the desired equivalence follows.

As for the “only if” part, since the non-uniform case is completely analogous, we may focus on the uniform case. Let $\varepsilon > 0$ be given, and take $\delta > 0$ as in “ f has uniformly at most one minimum in (X, d) ” such that $\delta \leq \varepsilon/3$. Now if $x, y \in X$ are such that $f(x)$ and $f(y)$ are both smaller than $\inf f + \delta$, then

$$\begin{aligned} \rho(x, y) &= d'(x, y) + |g(x) - g(y)| \\ &\leq d(x, y) + |f(x) - f(y)| \\ &< \varepsilon/3 + 2\delta \\ &\leq \varepsilon. \end{aligned}$$

To conclude we show that

- if (X, d) is complete and bounded, and f is continuous, then (X, ρ) is complete;
- if (X, d) is totally bounded and f is uniformly continuous, then (X, ρ) is totally bounded.

Suppose first that (X, d) is complete and bounded, and that f , and hence g , is continuous. Let (x_n) be a Cauchy sequence in (X, ρ) ; it then also is a Cauchy sequence in (X, d') , because $d' \leq \rho$. But, since X is bounded, d and d' are similar metrics as above, so (X, d') is complete; whence there exists $x \in X$ such that $x_n \rightarrow x$ in (X, d') as $n \rightarrow \infty$. Since g is continuous (with respect to d and thus also with respect to d'), we have

$$\rho(x, x_n) = d'(x, x_n) + |g(x) - g(x_n)| \rightarrow 0$$

as $n \rightarrow \infty$. Thus $x_n \rightarrow x$ in (X, ρ) as $n \rightarrow \infty$, so (X, ρ) is complete.

Suppose next that (X, d) is totally bounded and that f , and hence g , is uniformly continuous. Again d and d' are similar metrics, so (X, d') is totally bounded. Fix $\varepsilon > 0$ and let $\delta \in (0, \varepsilon/2)$ be such that for all $x, y \in X$, if $\rho(x, y) < \delta$, then $|g(x) - g(y)| < \varepsilon/2$. Let $\{x_1, \dots, x_n\}$ be a δ -approximation to (X, d') . Let $x \in X$, and pick $1 \leq i \leq n$ such that $d'(x, x_i) < \delta$. Then

$$\begin{aligned} \rho(x, x_i) &= d'(x, x_i) + |g(x) - g(x_i)| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

Hence $\{x_1, \dots, x_n\}$ is an ε -approximation to (X, ρ) . □

From what has been said before, it is immediate that if a subset S of a metric space X is complete and has uniformly at most one best approximation to a point x in X , then there is a (necessarily unique) best approximation to x in S .

We can now classify the counterpart of **UAM** for best approximations:

UAB *If a compact subset S of a metric space X has at most one best approximation to a point x of X , then S has uniformly at most one best approximation to x .*

Corollary 2 **UAB** is equivalent to **FT**.

Proof It is clear that **UAM** implies **UAB**, whereas the converse implication follows from Theorem 1. Since **UAM** is equivalent to **FT** [49], so is **UAB**. □

3 Finite-dimensional subspaces

We now turn our attention to a fundamental result in constructive approximation theory [17]. Let X be a normed space, let $x \in X$, and let S be a located subset of X . It is clear that S has uniformly at most one best approximation to x precisely when for each $\varepsilon > 0$ there exists $\delta > 0$ such that, for all $y, y' \in S$, if

$$\max \{ \|x - y\|, \|x - y'\| \} < \text{dist}(x, S) + \delta,$$

then $\|y - y'\| < \varepsilon$, whereas S has at most one best approximation to x if and only if

$$\max \{ \|x - y\|, \|x - y'\| \} > \text{dist}(x, S)$$

for all distinct $y, y' \in S$. While the latter notion of (non-uniform) uniqueness was coined in [17], a located subset S was called *quasi-proximal* in [18] if every $x \in X$ with at most one best approximation in S has a, necessarily unique, best approximation in S .

Finitely many vectors e_1, \dots, e_n in X are *linearly independent* if $\lambda_1 e_1 + \dots + \lambda_n e_n \neq 0$ whenever $|\lambda_1| + \dots + |\lambda_n| > 0$. A subspace E of X is *finite-dimensional* [17] if there exist linearly independent vectors $e_1, \dots, e_n \in E$ and bounded linear functionals $u_i : E \rightarrow \mathbf{R}$ ($1 \leq i \leq n$), the *component functions*, such that

$$x = \sum_{i=1}^n u_i(x) e_i$$

for all x in E ; in this case n is called the *dimension* of E , which of course is uniquely determined by E . Any finite-dimensional subspace E of X is located [17, Proposition 2.1] and complete [14, p. 307]; in particular, E is a closed subset of X .¹¹ We denote by $\mathbf{R}e$ the one dimensional subspace spanned by a nonzero element e of X . A subset S of X is said to be *convex* if $ty + (1 - t)y' \in S$ whenever $y, y' \in S$ and $t \in [0, 1]$.

Theorem 3 *Each finite-dimensional subspace of a normed space is quasi-proximal.*

In the following we give a dissection of Bridges' proof of Theorem 3 above; see Theorem 2.2 of [17], the (only) Theorem of [18], and Chapter 7, Theorem 2.12 of [14]. We do this with three goals in mind:

- to recast the proof in terms of uniform uniqueness;
- to elucidate the structure of the proof, and hence the structure of the inherent algorithm;

¹¹In [14] locatedness and closedness are included into the definition of a subspace.

— to further illustrate the relationship between minima and best approximations.

Let X be a normed space, let $x \in X$, let E be a finite-dimensional subspace of X , and let S be a located convex subset of E . For $\delta > 0$ we define

$$S_\delta = \{e \in S : \|x - e\| < \text{dist}(x, S) + \delta\}.$$

Then for each positive δ , S_δ is inhabited and—as the intersection of the convex set S with an open ball—bounded and convex. If $\delta_1 > \delta_2 > 0$, then $S_{\delta_2} \subset S_{\delta_1}$. We can rewrite the condition that S has uniformly at most one best approximation to x as

$$\forall \varepsilon > 0 \exists \delta > 0 \forall y, y' \in S_\delta (\|y - y'\| < \varepsilon).$$

We begin with, a slight extension of, the 1-dimensional case of Theorem 3, where the presence of a total ordering makes life much easier.

Lemma 4 *Let X be a normed space, let $x, e \in X$ with $e \neq 0$, and let S be a located convex subset of $\mathbf{R}e$. If x has at most one best approximation in S , then x has uniformly at most one best approximation in S .*

Proof Fix $\varepsilon > 0$, replace e by $e/\|e\|$, and let $t_1 < \dots < t_n$ be real numbers such that $\{t_1e, \dots, t_ne\}$ is an $\varepsilon/5$ -approximation to S . If $n \in \{1, 2\}$, then, since $\|e\| = 1$, the diameter of S_1 is less than ε and we need only set $\delta = 1$; therefore we may assume that $n > 2$. Since x has at most one best approximation in S , there exists $\delta > 0$ and $i_0 \in \{1, \dots, n\}$ such that $\|x - t_ie\| > \delta$ for all $i \neq i_0$. Let $t \in S_\delta$; then $t \neq t_i$ for all $i \neq i_0$. Given that S_δ is nonempty and convex, we must have that $\|te - t_1e\| < \varepsilon/5$, $\|te - t_ne\| < \varepsilon/5$, or there exists $j \neq i_0$ such that $S_\delta \subset (t_j, t_{j+2})e$. In the first case, $S_\delta \subset (-\infty, t_3)e \cap S$ which has diameter less than ε since $\|e\| = 1$ and by our choice of t_1, \dots, t_n ; similarly, in the second case $S_\delta \subset (t_{n-2}, \infty)e \cap S$, so $\text{diam}(S) < \varepsilon$. In the final case $\text{diam}(S) < 2\varepsilon/5$. Hence in all cases, for all $y, y' \in S_\delta$, $\|y - y'\| < \varepsilon$. \square

Corollary 5 *Let X be a normed space, let $x, e \in X$ with $e \neq 0$, and let S be a closed located convex subset of $\mathbf{R}e$. If x has at most one best approximation in $\mathbf{R}e$, then x has a best approximation in S .*

Extending this to encompass Bridges' main lemma [18] seems to require countable choice, at least in the weak form given in [23] that is classically valid without any choice.

Lemma 6 *Let X be a normed space, let $x, e \in X$ with $e \neq 0$, let S be a closed located convex subset of $\mathbf{R}e$, and let $d \in \mathbf{R}$ be such that*

$$\max \{\|x - s\|, \|x - s'\|\} > d$$

for all distinct $s, s' \in S$. Then there exists $\tau \in \mathbf{R}$ such that $\tau e \in S$ and if $\|x - \tau e\| > d$, then $\text{dist}(x, S) > d$.

Proof We first note that $d \leq \text{dist}(x, S)$. Let (λ_n) be an increasing binary sequence such that

$$\begin{aligned}\lambda_n = 0 &\Rightarrow \text{dist}(x, S) - d < 1/n, \\ \lambda_n = 1 &\Rightarrow \text{dist}(x, S) - d > 1/(n+1),\end{aligned}$$

and, using Lemma 4, construct a sequence (δ_n) of positive real numbers such that $\text{diam}(S_{\delta_n}) < 1/(n+1)$ for each n . We construct a Cauchy sequence (t_n) in \mathbf{R} as follows. If $\lambda_n = 0$, pick $t \in S_{\delta_n}$ and set $t_n = t$; if $\lambda_n = 1$ set $t_n = t_{n-1}$. Let τ be the limit of (t_n) in \mathbf{R} . Since S is closed, we have $\tau e \in S$. Suppose that $\|x - \tau e\| > d$ and let $N > 0$ be such that

$$\|x - \tau e\| > d + 2/N.$$

If $\lambda_N = 0$, then $\|x - \tau e\| < \text{dist}(x, S) + 1/N$ and $\text{dist}(x, S) - d < 1/N$, so

$$\|x - \tau e\| < \text{dist}(x, S) + 1/N < d + 2/N.$$

This contradiction ensures that $\lambda_N = 1$; whence $\text{dist}(x, S) > d$. \square

The particular case $d = \text{dist}(x, \mathbf{R}e)$ of Lemma 6 says that if S has at most one best approximation to x , then it does have a—necessarily unique—best approximation.

In [17], Bridges proceeds to prove Theorem 3 by an induction on the dimension of the finite-dimensional subspace. The next proposition isolates this induction. (The previous extension to closed located convex subsets of the finite-dimensional subspaces breaks down here; a counterexample can be constructed by extending that, in [17], of a linear subspace without a best approximation.) We need a few more definitions, and a lemma from [24]. Let E be a closed located subspace of a normed space X . We define a semi-norm $\|\cdot\|_{X/E}$ on X by

$$\|x\|_{X/E} = \rho(x, E).$$

With the inequality given by

$$x \neq y \Leftrightarrow \|x - y\|_{X/E} \neq 0$$

$(X, \|\cdot\|_{X/E})$ becomes a normed space called the *quotient space* of X by E , written X/E ; we read X/E as X modulo E . The following is Lemma 4.2.2 of [24]:

Lemma 7 Let $\{e_1, \dots, e_n\}$ be a basis for an n -dimensional subspace V of a normed space X , let $1 \leq m < n$, and let W be the subspace of X with basis $\{e_1, \dots, e_m\}$. Then the span of $\{e_{m+1}, \dots, e_n\}$ is an $(n - m)$ -dimensional subspace of the quotient space X/W .

Proposition 8 Let X be a normed space, let E be a finite-dimensional subspace of X , and let $x \in X$ be such that x has at most one best approximation in E . Then for each finite-dimensional subspace E' of E , x has at most one best approximation in E/E' (viewed as a subspace of X/E').

Proof We proceed by induction on the dimension of E . If $E = \mathbf{R}e$, then the result is trivial. Now suppose that we have proved the result for subspaces with dimension $n - 1$, let $\{e_1, \dots, e_n\}$ be a basis for E , and let E' be a finite-dimensional linear subspace of E . Without loss of generality $e_1 \in E'$.

Let $v_1, v_2 \in E$ be such that $\|v_1 - v_2\|_{X/\mathbf{R}e_1} > 0$. If $t, t' \in \mathbf{R}$ with $t \neq t'$, then for $i = 1, 2$

$$\|(v_i - te_1) - (v_i - t'e_1)\| = |t - t'| \|e_1\| \neq 0,$$

so

$$\max\{\|x - (v_i - te_1)\|, \|x - (v_i - t'e_1)\|\} > \text{dist}(x, E).$$

Applying Lemma 6 with $x = x - v_i, e = e_1$, and $d = \text{dist}(x, E)$ produces τ_i such that if $\|x - (v_i - \tau_i e_1)\| > \text{dist}(x, E)$, then $\text{dist}(x - v_i, \mathbf{R}e_1) > \text{dist}(x, E)$.

Since x has at most one best approximation in E , either

$$\|x - v_1 - \tau_1 e_1\| > \text{dist}(x, E) \text{ or } \|x - v_2 - \tau_2 e_1\| > \text{dist}(x, E).$$

Hence

$$\max\{\text{dist}(x - v_1, \mathbf{R}e_1), \text{dist}(x - v_2, \mathbf{R}e_1)\} > \text{dist}(x, E),$$

so x has at most one best approximation in E modulo $\mathbf{R}e_1$. Since, by Lemma 7, $E/\mathbf{R}e_1$ is an $(n - 1)$ -dimensional subspace of $X/\mathbf{R}e_1$, it follows from our induction hypothesis that x has at most one best approximation in E modulo $\mathbf{R}e_1$ modulo E' . Since $X/E' = (X/\mathbf{R}e_1)/E'$, this completes the induction. \square

We can now give the uniform uniqueness version of Theorem 3, from which that theorem follows.

Theorem 9 Let X be a normed space, let E be a finite-dimensional subspace of X , and let $x \in X$ be such that x has at most one best approximation in E . Then x has uniformly at most one best approximation in E .

Proof Let $\varepsilon > 0$ and let e_1, \dots, e_n be a basis for E with component functions u_1, \dots, u_n . Since all norms on a finite-dimensional subspace are equivalent (Corollary 4.1.9 of [24]), it suffices to show that there exists $\delta > 0$ such that

$$|u_i(y) - u_i(y')| < \varepsilon$$

for all $y, y' \in S_\delta$ and each $i \in \{1, \dots, n\}$.

For each $i \in \{1, \dots, n\}$ let

$$E_i = \text{span}\{e_j : 1 \leq j \leq n, j \neq i\}.$$

By Proposition 8, x has at most one best approximation in E modulo E_i for each i . Using Lemma 4, construct $\delta_1, \dots, \delta_n$ such that if

$$\|x - te_i\|_{X/E_i} < \text{dist}_{X/E_i}(x, \mathbf{R}e_i) + \delta_i,$$

then $|t - t'| < \varepsilon$. It remains to set $\delta = \min\{\delta_i : 1 \leq i \leq n\}$. □

4 Convex subsets

We finally review the case of best approximations to a convex set in a uniformly convex normed space. In all the sources referred to in this section the following definition, a uniform-quantitative variant of the one of a strictly convex normed space, has been put in one way or the other: a normed space X is *uniformly convex* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|(x + y)/2\| > 1 - \delta \Rightarrow \|x - y\| < \varepsilon$$

for all $x, y \in X$ with $\|x\| = \|y\| = 1$. (In [36] the dependence of δ on ε is expressed as a modulus of uniform convexity.)

With Chapter 9, Exercise 5 of [13] and again with Chapter 7, Exercise 11 of [14], which in this section will be referred to as *Bishop's exercise*, it is claimed that the following statement has a constructive proof: if S is a closed located convex subset of a uniformly convex Banach space X , then each point in X has a unique best approximation in S . (Proposition 17.5 of [36] says the same, with a modulus of uniform convexity for X .) Note that there is no need to suppose that the uniformly convex normed space X be complete: it suffices to assume that the convex located subset S is complete.

Since finite-dimensional (linear) subspaces are complete located convex subsets (see above), Theorem 3.1 of [17] is a special case of Bishop's exercise. This Theorem 3.1 says that if E is a finite-dimensional subspace of a uniformly convex Banach space X ,

then each point in X has a, necessarily unique, best approximation in E . It was proved by what then was a new technique involving uniqueness; more precisely Theorem 3.1 was reduced to Theorem 2.2 of [17], a forerunner of the quasi-proximality of finite-dimensional subspaces of arbitrary normed spaces (see Theorem 3 above). An inspection of the very proof of this Theorem 3.1 reveals that each point in a uniformly convex normed space X has indeed uniformly at most one best approximation in any given finite-dimensional subspace E of X .

The content of Theorem 6 of [22] is that if E is a complete located subspace of a uniformly convex normed space X , then each point of X has a unique best approximation in E . Inasmuch as E need not be finite-dimensional, this clearly generalises Theorem 3.1 of [17].

When going through the proof of Theorem 6 of [22] one will notice that E may even be a convex subset rather than a (linear) subspace; whence one might think that this theorem is nothing but a remake of Bishop's exercise. The point of Theorem 6 of [22] is, however, that its proof is done without countable choice, again with a concept of a completion that is free of sequences; on the other hand it is clear that in the intended proof of Bishop's exercise a sequence would have to be chosen to approximate the best approximation.

Last but not least, by looking at the proof of Theorem 6 of [22] together with its Lemma 5 we can exhibit the essence of Bishop's exercise: if S is a convex located subset of a uniformly convex normed space X , then each point in X has uniformly at most one best approximation in S . (This is Proposition 17.4 of [36], again with moduli of uniform convexity and uniqueness.)

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