Cardinal Invariants of Strongly Porous Sets

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Abstract: In this work we study cardinal invariants of the ideal **SP** of strongly porous sets on $^\omega 2$. We prove that $\mathsf{add}(\mathbf{SP}) = \omega_1$, $\mathsf{cof}(\mathbf{SP}) = \mathfrak{c}$ and that it is consistent that $\mathsf{non}(\mathbf{SP}) < \mathsf{add}(\mathcal{N})$, answering questions of Hrušák and Zindulka. We also find a connection between strongly porous sets on $^\omega 2$ and the Martin number for σ-linked partial orders, and we use this connection to construct a model where all the Martin numbers for σ-k-linked forcings are mutually different.

1   Introduction

In 1967, Dolženko [5] began the study of σ-porous sets[1] and since then, many applications have been found. One of these appears in [10], where Preiss and Zajíček proved that, given a Banach space X with a separable dual and a continuous convex function f on X, the set of points in which f is not Fréchet differentiable is σ-porous. Other applications can be found in Belna, Evans and Humke [2], Lindenstrauss and Preiss [7], Lindenstrauss, Preiss and Tišer [8], Reich and Zaslavski [11], Renfro [12] and Zaslavski [19].

We shall study the notion of strong porosity: given a metric space $\langle X, d \rangle$, a subset $A \subseteq X$ is *strongly porous* if there is a $p > 0$ such that for any $x \in X$ and any $0 < r < 1$, there is $y \in X$ such that $B_{p \cdot r}(y) \subseteq B_r(x) \setminus A$. In this paper we will refer to strongly porous sets as porous sets. We shall work mostly with porous sets in $^\omega 2$: we say that a set $A \subseteq {}^\omega 2$ is n-porous if for every $s \in {}^{<\omega}2$ there is a $t \in {}^n 2$ such that $\langle s^\frown t \rangle \cap A = \emptyset$. By $s^\frown t$ we denote the concatenation of s followed by t, and by $\langle s \rangle$ we denote the cone of s, that is $\langle s \rangle = \{ f \in {}^\omega 2 : s \subseteq f \}$. It is easy to see (see Hrušák and Zindulka [6]) that

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[1]A σ-(upper/lower) porous set is the countable union of (upper/lower) porous sets

a set $A \subseteq {}^\omega 2$ is porous if and only if there is an $n \in \omega$ such that A is n-porous. Zajíček [17] proved that a set A in a metric space $\langle X, d \rangle$ is σ-porous if and only if it is σ-lower porous, where A is *lower porous* if for every $x \in X$ there exist $\rho_x > 0$ and $r_x > 0$ such that for any $0 \leq r \leq r_x$ there is $y \in X$ such that $B_{\rho_x \cdot r}(y) \subseteq B_r(x) \setminus A$. Another classical notion of porosity is upper porosity: a set A in a metric space $\langle X, d \rangle$ is *upper porous* if for every $x \in X$ there is $\rho_x > 0$ and a sequence $r_n \to 0$ such that for every $n \in \omega$ there is $y_n \in X$ such that $B_{\rho_x \cdot r_n}(y_n) \subseteq B_{r_n}(x) \setminus A$. It is easy to see that lower porosity implies upper porosity.

We will denote the σ-ideal[2] generated by porous sets on ${}^\omega 2$ by **SP**, the σ-ideal generated by n-porous sets by $\mathbf{SP}_n$, and the σ-ideal generated by upper porous sets by **UP**. Observe that $\mathbf{SP}_1$ is the ideal of countable sets of ${}^\omega 2$. Further research about different types of porosity can be found in Rojas-Rebolledo [16], Zajíček [17], Zapletal [18], Zelený [20] and Zelený and Zajíček [21].

Cardinal invariants of these σ-ideals have been studied in Brendle [3], Hrušák and Zindulka [6] and Repický [13, 14, 15]. Recall that, given a σ-ideal $\mathcal{I}$ over a set X, the following are the standard cardinal invariants of $\mathcal{I}$:

$$
\begin{aligned}
\mathsf{add}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup \mathcal{A} \notin \mathcal{I}\} \\
\mathsf{cov}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup \mathcal{A} \neq X\} \\
\mathsf{non}(\mathcal{I}) &= \min\{|Y| : Y \subseteq X \wedge Y \notin \mathcal{I}\} \\
\mathsf{cof}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \forall B \in \mathcal{I} \, (\exists A \in \mathcal{A} \, (B \subseteq A))\}
\end{aligned}
$$

In [6], Hrušák and Zindulka proved that the cardinal invariants of the σ-ideal of lower porous sets in the real line are the same as the cardinal invariants of **SP**. They proved that $\mathsf{non}(\mathbf{SP}) < \mathfrak{m}_{\sigma\text{-centered}}$ is consistent, that $\mathsf{cov}(\mathbf{SP}) > \mathsf{cof}(\mathcal{N})$ is consistent, and that $\mathsf{cov}(\mathbf{SP}) < \mathfrak{c}$ is consistent, where $\mathfrak{m}_{\sigma\text{-centered}}$ is the smallest cardinal where the Martin's axiom for σ-centered forcings fails and $\mathcal{N}$ is the ideal of sets of Lebesgue measure zero.

There are some analogous inequalities that hold for **UP**. In [15], M. Repický proved that $\mathsf{non}(\mathbf{UP}) \geq \mathfrak{m}_{\sigma\text{-centered}}$ and $\mathsf{cov}(\mathbf{UP}) \leq \mathsf{cof}(\mathcal{N})$ holds. He proved in [13] that $\mathsf{non}(\mathbf{UP}) \geq \mathsf{add}(\mathcal{N})$ and in [3], J. Brendle proved that $\mathsf{add}(\mathbf{UP}) = \omega_1$ and $\mathsf{cof}(\mathbf{UP}) = \mathfrak{c}$ hold. In [6], Hrušák and Zindulka asked if the last three inequalities hold also for the **SP** ideal. In this work we show that $\mathsf{add}(\mathbf{SP}) = \omega_1$, $\mathsf{cof}(\mathbf{SP}) = \mathfrak{c}$ and that it is consistent that $\mathsf{non}(\mathbf{SP}) < \mathsf{add}(\mathcal{N})$.

Given $k \in \omega$ and a forcing notion $\mathbb{P}$ a subset $A \subseteq \mathbb{P}$ is *k-linked* if for every collection $\{a_i : i \in k\}$ of k elements of A, there is an $a \in \mathbb{P}$ such that for every $i \in k$, $a \leq a_i$. $\mathbb{P}$

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[2] σ-ideals are ideals closed under countable unions of their elements.

is σ-k-*linked* if $\mathbb{P}$ is the countable union of k-linked subsets of $\mathbb{P}$. We will denote by $\mathfrak{m}_k$ the *Martin number for* σ-k-linked forcings, that is, the smallest cardinal κ such that there is a σ-k-linked forcing $\mathbb{P}$ and κ many $\mathbb{P}$-dense subsets of $\mathbb{P}$ such that no filter of $\mathbb{P}$ intersects them all. In [4], Brendle and Shelah constructed a model where all the Martin numbers of the form $\mathfrak{m}_{2^i}$ are distinct. We will show a connection between σ-k-linked forcings and porous sets and we will use this connection to construct a model where all the Martin numbers $\mathfrak{m}_i$ are different all at once.

If X, Y are sets, then ${}^Y X$ is the set of all functions from Y to X and ${}^{<\omega}X = \bigcup_{n\in\omega} {}^n X$. If $\sigma, s \in {}^{<\omega}X$, then we will denote that σ is an initial segment of s by $\sigma \sqsubseteq s$. A set $T \subseteq {}^{<\omega}X$ is a tree if it is closed under restrictions to initial segments. If $T \subseteq {}^{<\omega}X$ is a tree, then by $[T]$ we denote the set of branches of T, that is, $[T] = \{f \in {}^\omega X : \forall n \in \omega\ (f{\restriction}n \in T)\}$. With $\mathsf{end}(T)$ we will denote the end nodes of T, that is the nodes of T without extensions. In our forcing notation, the stronger conditions are the smaller ones. In general, our notation follows Bartoszyński and Judah [1].

2 Additivity and cofinality

The main goal of this section is to prove that $\mathsf{add}(\mathbf{SP}) = \omega_1$ and $\mathsf{cof}(\mathbf{SP}) = \mathfrak{c}$. We will use the following notion.

Definition 2.1 Let $k \in \omega$. A tree $T \subseteq {}^{<\omega}2$ is a k-porous tree *if for every* $s \in {}^{<\omega}2$ *there is* $t \in {}^k 2$ *such that* $s^\frown t \notin T$.

Note that $A \subseteq {}^\omega 2$ is k-porous if and only if there is a k-porous tree T such that $[T]$ contains A.

Theorem 2.2 *There is a family* $\{T_f : f \in {}^\omega 2\}$ *of 2-porous trees such that for every* $X \in \mathbf{SP}$, *the set* $\{f \in {}^\omega 2 : [T_f] \subseteq X\}$ *is countable.*

Proof We will construct the family $\{T_f : f \in {}^\omega 2\}$ as follows: For every $a \subseteq {}^{<\omega}2$ such that $|a| = 2^n$, let $\varphi_a : a \to {}^n 2$ be a bijective function. For every $i \in \omega$, let $\psi_i : \{a \subseteq {}^i 2 : \exists k \in \omega\ (|a| = 2^k)\} \to \omega \setminus \{0\}$ be an injective function. If $a \subseteq {}^i 2$ and $|a| = 2^k$, define

$$\sigma_a = \langle 0, \underbrace{1, \ldots, 1}_{2\psi_i(a)\text{ times}}, 0 \rangle.$$

For each $\sigma \in {}^{<\omega}2$, we will recursively define a finite tree T_σ as follows: $T_\emptyset = \{\emptyset\}$, and if T_σ is defined then

$$
\begin{aligned}
T_{\sigma^\frown i} = & \{s \in {}^{<\omega}2 : \exists t \in \mathsf{end}(T_\sigma)\, (\exists j \in \omega\, (\exists a \subseteq {}^{|\sigma|+1}2 \\
& (|a| = 2^j \wedge \sigma^\frown i \in a \wedge s \sqsubseteq t^\frown \sigma_a^\frown \varphi_a(\sigma^\frown i))))\} \\
& \cup \{s \in {}^{<\omega}2 : \exists t \in \mathsf{end}(T_\sigma)\, (s \sqsubseteq t^\frown \langle 1, 1\rangle)\}.
\end{aligned}
$$

It is easy to see that, for each $\sigma \in {}^{<\omega}2$, T_σ is a finite 2-porous tree and that if $\sigma \sqsubseteq \tau$, then $T_\sigma \subseteq T_\tau$. For each $f \in {}^{\omega}2$, define $T_f = \bigcup_{n \in \omega} T_{f \restriction n}$. It follows easily that each T_f is a 2-porous tree.

We will show that the family $\{T_f : f \in {}^{\omega}2\}$ is the family we were looking for: Let $X \in \mathbf{SP}$. Without loss of generality we will assume that $X = \bigcup_{i \in \omega}[T_i]$, where T_i is an $i + 1$-porous tree. We must show that the set $B = \{f \in {}^{\omega}2 : [T_f] \subseteq X\}$ is countable: For each $s, t \in {}^{<\omega}2$ and each $n \in \omega$, define $B_{s,t,n} = \{f \in {}^{\omega}2 : t \sqsubseteq f, s \in T_t \wedge [T_f] \cap \langle s \rangle \subseteq [T_n]\}$. We will see that $B \subseteq \bigcup_{s,t \in {}^{<\omega}2, n \in \omega} B_{s,t,n}$: If f is such that $f \in B$, then $[T_f] \subseteq \bigcup_{n \in \omega}[T_n]$. Using the Baire Category Theorem we can find $s \in T_f$ and $n \in \omega$ such that $[T_f] \cap \langle s \rangle \subseteq [T_n]$. Find $k \in \omega$ such that $s \in T_{f \restriction k}$. It follows that $f \in B_{s, f \restriction k, n}$. To finish the proof we will see that each $B_{s,t,n}$ has at most $2^{n+1} - 1$ elements: Suppose this is not the case and let $s, t \in {}^{<\omega}2$, $n \in \omega$ and $\{f_i\}_{i < 2^{n+1}} \subseteq B_{s,t,n}$. Extend s to σ such that $\sigma \in \mathsf{end}(T_t)$. Let $j \in \omega$ be such that the set $a = \{f_i \restriction j : i < 2^{n+1}\}$ has 2^{n+1} elements and let

$$
s_0 = \sigma^\frown \underbrace{\langle 1, \dots, 1\rangle}_{2 \cdot (j - |t| - 1) \text{ times}}{}^\frown \sigma_a.
$$

The tree T_n is $n + 1$-porous, so there is a $\tau \in 2^{n+1}$ such that $s_0^\frown \tau \notin T_n$. Find $k < 2^{n+1}$ such that $\varphi_a(f_k \restriction j) = \tau$ and observe that $s_0^\frown \tau = s_0^\frown \varphi_a(f_k \restriction j) \in T_{f_k}$. As a consequence, $[T_{f_k}] \cap \langle s \rangle \nsubseteq [T_n]$, but this contradicts the fact that $f_k \in B_{s,t,n}$. This implies that each $B_{s,t,n}$ is finite, and therefore B is countable. $\qquad\square$

We can now prove the main result of this section.

Corollary 2.3 $\mathsf{add}(\mathbf{SP}) = \omega_1, \mathsf{cof}(\mathbf{SP}) = \mathfrak{c}$.

Proof Let $\{T_f : f \in {}^{\omega}2\}$ be the family given by the theorem above. If $H \subseteq {}^{\omega}2$ is an uncountable set, then $\bigcup\{[T_f] : f \in H\} \notin \mathbf{SP}$. As a consequence, $\mathsf{add}(\mathbf{SP}) = \omega_1$. On the other hand, if $\kappa < \mathfrak{c}$ and if $\{X_\alpha : \alpha < \kappa\} \subseteq \mathbf{SP}$, then there is an $f \in {}^{\omega}2$ such that, for every $\alpha < \kappa$, $[T_f] \nsubseteq X_\alpha$ and therefore $\mathsf{cof}(\mathbf{SP}) = \mathfrak{c}$. $\qquad\square$

Observe that this last proof can be used to show that $\mathsf{add}(\mathbf{SP}_n) = \omega_1 = \mathsf{add}(\mathbf{SP})$ and $\mathsf{cof}(\mathbf{SP}_n) = \mathfrak{c} = \mathsf{cof}(\mathbf{SP})$.

3   Uniformity number

In this section we will study some properties concerning the uniformity number of porosity ideals. We will prove the consistency of $\mathsf{non}(\mathbf{SP}) < \mathsf{add}(\mathcal{N})$. We will also develop some tools that we will use later in the paper. We will need the following concept, inspired by the concept of a k-Sacks tree in $^{<\omega}k$.

Definition 3.1 *Let $k > 1$. A tree $T \subseteq {}^{<\omega}k$ is a k-anti-Sacks tree if for every $s \in T$ there is $i < k$ such that $s^\frown\langle i \rangle \notin T$. We will denote by $\mathbf{AS}_k$ the σ-ideal generated by the branches of k-anti-Sacks trees.*

This notion is the analogue of the notion of 1-porous tree in $^{<\omega}k$ and it is closely related to the k-Sacks forcing. Recall that a *k-Sacks tree T* is a tree on $^{<\omega}k$ such that for every $s \in T$, there is a $t \in T$ such that, for every $i < k$, $t^\frown i \in T$. The *k-Sacks forcing $\mathbb{S}_k$* is the collection of all k-Sacks trees ordered by reverse inclusion. It is well-known that the k-Sacks forcing is equivalent to $\mathrm{Borel}(^{\omega}k)/\mathbf{AS}_k$ (see Newelski and Rosłanowski [9]).

Using a similar argument to the one we gave in the last section, it is possible to show that $\mathsf{add}(\mathbf{AS}_k) = \omega_1$ and that $\mathsf{cof}(\mathbf{AS}_k) = \mathfrak{c}$. Alternatively, a proof of this fact can be found in [9].

The ideals $\mathbf{SP}_k$ and $\mathbf{AS}_{2^k}$ share many properties. Many of the results in this work will concern properties of the ideals $\mathbf{AS}_k$ that are also valid for the ideals $\mathbf{SP}_k$, and the proofs for both ideals are almost the same. Whenever we state a property about one of these ideals that is also valid for the other one, we will only give the proof for the ideal $\mathbf{AS}_k$.

We shall introduce a notion that we will use to keep the uniformity number small in a forcing extension.

Definition 3.2 *Let $\mathbb{P}$ be a forcing notion and let $A \subseteq {}^{\omega}k$ $(A \subseteq {}^{\omega}2)$ be such that $A \notin \mathbf{AS}_k$ $(A \notin \mathbf{SP}_k)$. We say that $\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{AS}_k)$ in A ($\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{SP}_k)$ in A) if for every $\mathbb{P}$-name $\dot{X}$ of a k-anti-Sacks tree (k-porous tree) there is a $Y \in \mathbf{AS}_k$ ($Y \in \mathbf{SP}_k$) such that, for every $x \in A$, if $x \notin Y$ then $\mathbb{P} \Vdash$ "$x \notin [\dot{X}]$". We will say that $\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{AS}_k)$ ($\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{SP}_k)$) if $\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{AS}_k)$ in ${}^{\omega}k$ ($\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{SP}_k)$ in ${}^{\omega}2$).*

It is easy to see that, if $\mathbb{P}$ strongly preserves $\mathrm{non}(\mathbf{AS}_k)$ in A, then $\mathbb{P} \Vdash$ "$A \notin \mathbf{AS}_k$" and if $\mathbb{P}$ strongly preserves $\mathrm{non}(\mathbf{AS}_k)$, then $\mathbb{P}$ strongly preserves $\mathrm{non}(\mathbf{AS}_k)$ in A for every $A \subseteq {}^\omega k$. The following lemma is easy to prove.

Lemma 3.3 *Suppose that a forcing notion $\mathbb{P}$ strongly preserves $\mathrm{non}(\mathbf{SP}_k)$ for every $k \in \omega$, then $\mathbb{P} \Vdash$ "${}^\omega 2 \cap V \notin \mathbf{SP}$".*

Proof The proof is straightforward from the definitions. □

The next lemma shows that there is a connection between anti-Sacks trees and σ-k-linked forcings.

Lemma 3.4 *Let $\mathbb{P}$ be a forcing notion. If $\mathbb{P}$ is σ-k-linked, then $\mathbb{P}$ strongly preserves $\mathrm{non}(\mathbf{AS}_k)$.*

Proof Let $\{\mathbb{P}_i : i \in \omega\}$ be a sequence of k-linked subsets such that $\mathbb{P} = \bigcup_{i \in \omega} \mathbb{P}_i$. Let $\dot{A}$ be a $\mathbb{P}$-name of a k-anti-Sacks tree. Define $T_n \subseteq {}^\omega k$ as follows:

$$T_n = \left\{ s \in {}^{<\omega}k : \exists p \in \mathbb{P}_n (p \Vdash \text{``}s \in \dot{A}\text{''}) \right\}$$

We claim that, for each $n \in \omega$, T_n is a k-anti-Sacks tree. Suppose this is not the case, so there is an $s \in T_n$ such that, for every $i \in k$, $s^\frown i \in T_n$. For every $i \in k$, we can pick a condition $p_i \in \mathbb{P}_n$ such that $p_i \Vdash \text{``}s^\frown i \in \dot{A}\text{''}$. Let $p \in \mathbb{P}$ be such that, for every $i \in k$, $p \leq p_i$. Then $p \Vdash \text{``}\forall i \in k(s^\frown i \in \dot{A})\text{''}$. This contradicts the fact that $\dot{A}$ is a $\mathbb{P}$-name of a k-anti-Sacks tree.

To conclude the proof, note that for every $x \in {}^\omega k$, if $p \Vdash \text{``}x \in [\dot{A}]\text{''}$, then $x \in [T_n]$, where n is such that $p \in \mathbb{P}_n$. □

The lemma above is optimal in the sense that, for each k, there is a σ-$(k-1)$-linked forcing $\mathbb{P}_k$ such that $\mathbb{P}_k \Vdash$ "${}^\omega k \cap V \in \mathbf{AS}_k$" and therefore $\mathbb{P}_k$ does not strongly preserve $\mathbf{AS}_k$. This will be proved in the next section. There is also a relation between porous sets in ${}^\omega 2$ and σ-k-linked forcings.

Lemma 3.5 *Let $\mathbb{P}$ be a forcing notion. If $\mathbb{P}$ is σ-2^k-linked, then $\mathbb{P}$ strongly preserves $\mathrm{non}(\mathbf{SP}_k)$.*

Proof The proof is similar to the previous lemma. □

We shall show that strong preservation of $\mathsf{non}(\mathbf{AS}_k)$ and $\mathsf{non}(\mathbf{SP}_k)$ is preserved by finite support iterations.

Lemma 3.6 *Let $A \subseteq {}^{\omega}k$ and $\mathcal{I} \in \{\mathbf{SP}_n, \mathbf{AS}_k : n > 0, k > 1\}$.*

(1) *If $\mathbb{P}$ is a forcing notion such that $\mathbb{P}$ strongly preserves $\mathsf{non}(\mathcal{I})$ in A and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a forcing such that $\mathbb{P} \Vdash$ "$\dot{\mathbb{Q}}$ strongly preserves $\mathsf{non}(\mathcal{I})$ in A", then $\mathbb{P} * \dot{\mathbb{Q}}$ strongly preserves $\mathsf{non}(\mathcal{I})$ in A.*

(2) *If $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha \leq \gamma \rangle$ is a finite support iteration of c.c.c. forcings such that $\mathbb{P}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha$ strongly preserves $\mathsf{non}(\mathcal{I})$ in A" for each $\alpha \in \gamma$, then $\mathbb{P}_\gamma$ strongly preserves $\mathsf{non}(\mathcal{I})$ in A.*

Proof We will only show the case when $\mathcal{I} = \mathbf{AS}_k$ as the other cases are similar. The part (1) is easy, we will proceed with part (2) by induction on γ. It is easy to see that the lemma holds for successor ordinals, and if γ has uncountable cofinality we can use a standard reflection argument to show that $\mathbb{P}_\gamma$ strongly preserves $\mathsf{non}(\mathbf{AS}_k)$ in A, so it is enough to show that the lemma holds for $\gamma = \omega$. Let $\dot{T}$ be a $\mathbb{P}_\omega$-name of a k-anti-Sacks tree. For each $n \in \omega$, let $\dot{T}_n$ be a $\mathbb{P}_n$-name for the following set:

$$\dot{T}_n = \{ s \in {}^{<\omega}k : \mathbb{P}_{(n,\omega)} \Vdash \text{``} s \in \dot{T} \text{''} \}$$

It is easy to see that each $\dot{T}_n$ is a $\mathbb{P}_n$ name for a k-anti-Sacks tree. Now we use that each $\mathbb{P}_n$ strongly preserves $\mathsf{non}(\mathbf{AS}_k)$ to find a family $\{T_i^j : i, j \in \omega\}$ such that, for each $n \in \omega$, if $x \in A$ and $x \notin \bigcup_{i \in \omega}[T_i^n]$, then $\mathbb{P}_n \Vdash$ "$x \notin [\dot{T}_n]$". It is easy to see that the set $Y = \bigcup \{[T_i^j] : i, j \in \omega\}$ is the set we are looking for. $\square$

We will now prove the consistency of $\mathsf{non}(\mathbf{SP}) < \mathsf{add}(\mathcal{N})$. For constructing the model we are looking for, we will use the amoeba forcing $\mathbb{A}$ in the following presentation:

$$\mathbb{A} = \{ B \in \mathrm{Borel}({}^{\omega}2) : \mu(B) > \frac{1}{2} \}$$

Here $\mathrm{Borel}({}^{\omega}2)$ represents the collection of Borel subsets of the Cantor space and μ is the standard Lebesgue measure over ${}^{\omega}2$. The order is given by $A \leq B$ if and only if $A \subseteq B$. The following lemma is well-known (see eg Bartoszyński and Judah[1]). We include the simple proof for the sake of completeness.

Lemma 3.7 *The amoeba forcing is σ-n-linked for every $n \in \omega$.*

Proof Let $n \in \omega$. For every clopen C in $^\omega 2$, define

$$\mathbb{A}_C = \{A \in \mathbb{A} : \mu(C \backslash A) < \frac{1}{n} \cdot (\mu(C) - \frac{1}{2})\}.$$

We will show that $\mathbb{A} = \bigcup\{\mathbb{A}_C : C \text{ is a clopen in } ^\omega 2\}$. Let $A \in \mathbb{A}$ and let $\varepsilon > 0$ such that $\mu(A) = \frac{1}{2} + \varepsilon$. Find an open set $U \subseteq {}^\omega 2$ such that $A \subseteq U$ and $\mu(U \backslash A) < \frac{\varepsilon}{n}$. Now find a clopen set $C \subseteq U$ such that $\mu(C) > \frac{1}{2} + \varepsilon$. Then

$$\mu(C \backslash A) < \mu(U \backslash A) < \frac{\varepsilon}{n} = \frac{1}{n} \cdot (\frac{1}{2} + \varepsilon - \frac{1}{2}) < \frac{1}{n} \cdot (\mu(C) - \frac{1}{2}).$$

Therefore $A \in \mathbb{A}_C$. Now we must show that, for every clopen set $C \subseteq {}^\omega 2$, the intersection K of an arbitrary family $\{A_j : j \in n\} \subseteq \mathbb{A}_C$ is an element of $\mathbb{A}$. This is a consequence of the following calculations:

$$\mu(C) \leq \mu(K) + \sum_{j \in n} \mu(C \backslash A_j) < \mu(K) + \frac{1}{n} \cdot (\sum_{j \in n} \mu(C) - \frac{1}{2}) = \mu(K) + \mu(C) - \frac{1}{2}.$$

As a consequence, $\frac{1}{2} < \mu(K)$. Therefore $K \in \mathbb{A}$. $\square$

We are ready to prove the main result of this section. The method of the proof was suggested to us by J. Brendle.

Theorem 3.8 *If* ZFC *is consistent, then* ZFC $+$ non($\mathbf{SP}$) $<$ add($\mathcal{N}$) *is consistent.*

Proof Start with a model V such that $V \models CH$. Let $\mathbb{P} = \{\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_2\}$ be a finite support iteration of the amoeba forcing. It follows from the lemmas above that $\mathbb{P}$ strongly preserves non($\mathbf{SP}_k$) for every $k \in \omega$ and therefore $\mathbb{P} \Vdash$ "$^\omega 2 \cap V \notin \mathbf{SP}$". As a consequence, we have that $V[G] \models$ non($\mathbf{SP}$) $= \omega_1$. It is a known fact (see [1]) that $V[G] \models$ add($\mathcal{N}$) $= \omega_2$, hence $V[G] \models$ non($\mathbf{SP}$) $<$ add($\mathcal{N}$). $\square$

4 Martin numbers of σ-k-linked forcings

It is easy to see that $\mathfrak{m}_2 \leq \mathfrak{m}_3 \leq \ldots$ and, for each $k > 1$, it is possible to get the consistency of $\mathfrak{m}_k < \mathfrak{m}_{k+1}$ by forcing with a finite support iteration of σ-$(k+1)$-linked forcings over a model of CH. In [4], Brendle and Shelah constructed a model where all the cardinals of the form $\mathfrak{m}_{2^k}$ are different. In this section we will construct a model where all the Martin numbers $\mathfrak{m}_i$ are different at the same time. In this model, the cardinals non($\mathbf{AS}_i$) will be different all at once (so will be the cardinals non($\mathbf{SP}_i$)). We will use the following forcing notions. Given $k > 2$ let

$$\mathbb{P}_k = \{\langle s, F\rangle : (a) \ s \text{ is a finite } k\text{-anti-Sacks tree of height ht}(s),$$
$$(b) \ F \in [^\omega k]^{<\omega}, \text{ and } \lceil F{\upharpoonright}\Delta_F\rceil \text{ is a finite } k\text{-anti-Sacks tree, and}$$
$$(c) \ s \subseteq \lceil F{\upharpoonright}\Delta_F + 1\rceil\},$$

where $F{\upharpoonright}k = \{f{\upharpoonright}k : f \in F\}$, $\lceil F\rceil = \{\sigma \in {}^{<\omega}k : \exists f \in F \ (\sigma \subseteq f)\}$ and $\Delta_F = \min\{n \in \omega : |F{\upharpoonright}n| = |F|\}$. The order is defined by $\langle s', F'\rangle \leq \langle s, F\rangle$ if and only if $s \subseteq s'$ and $F \subseteq F'$. This forcing notion will be used to work with the ideal $\mathbf{AS}_k$. For the ideal $\mathbf{SP}_k$, we will be using a similar forcing notion. Given $k > 1$ let

$$\mathbf{P}_k = \{\langle s, F\rangle : (a) \ s \text{ is a finite } k\text{-porous tree, and}$$
$$(b) \ F \in [^\omega 2]^{<\omega}, \text{ and } \lceil F{\upharpoonright}\Delta_F\rceil \text{ is a finite } k\text{-anti-Sacks tree, and}$$
$$(c) \ s \subseteq \lceil F{\upharpoonright}\Delta_F + 1\rceil\}.$$

The order is defined by $\langle s', F'\rangle \leq \langle s, F\rangle$ if and only if $s \subseteq s'$ and $F \subseteq F'$. We will be using the following proposition.

Proposition 4.1 *Given a $k > 2$ and an $i > 1$, $\mathbb{P}_k \Vdash ``^\omega k \cap V \in \mathbf{AS}_k$'' and $\mathbf{P}_i \Vdash ``^\omega 2 \cap V \in \mathbf{SP}_i$''.*

Proof We will only check that $\mathbb{P}_k \Vdash ``^\omega k \cap V \in \mathbf{AS}_k$'' as the other part is similar. It is easy to see that, for every $f \in {}^\omega k$ and $n \in \omega$, the following sets are dense in $\mathbb{P}$:

$$D_f = \{\langle s, F\rangle \in \mathbb{P}_k : \exists \sigma \in {}^{<\omega}k \ (\sigma^\frown f{\upharpoonright}(\omega \setminus |\sigma|) \in F)\}$$
$$E_n = \{\langle s, F\rangle \in \mathbb{P}_k : \Delta_F > n \wedge s = \lceil F{\upharpoonright}\Delta_F + 1\rceil\}$$

If $G \subseteq \mathbb{P}_k$ is a filter meeting all these dense sets, then, using that the sets E_n are dense, it follows that $T = \bigcup\{s : \exists F(\langle s, F\rangle \in G)\}$ is a k-anti-Sacks tree. If $\sigma \in {}^{<\omega}k$ and if $C[\sigma] = \{\sigma^\frown x{\upharpoonright}(\omega \setminus |\sigma|) : x \in [T]\}$, then, using that the D_f are dense, it follows that $^\omega k \cap V \subseteq \bigcup\{C[\sigma] : \sigma \in {}^{<\omega}k\} \in \mathbf{AS}_k$. $\qquad\square$

The last proposition together with the Lemma 3.4 implies that $\mathbb{P}_k$ is not σ-k-linked. In contrast with this last observation, we have the following proposition.

Proposition 4.2 *For each $k > 1$, $\mathbb{P}_{k+1}$ is σ-k-linked and $\mathbf{P}_k$ is σ-$(2^k - 1)$-linked.*

Proof Again, we will only check that $\mathbb{P}_{k+1}$ is σ-k-linked, the other part is done in a similar way: For any two finite $(k+1)$-anti-Sacks trees s, t of height $\mathsf{ht}(s), \mathsf{ht}(t)$ respectively, define

$$P(s, t) = \{\langle s, F\rangle \in \mathbb{P}_{k+1} : \mathsf{ht}(t) > \Delta_F \wedge F{\restriction}\mathsf{ht}(t) = t\}.$$

It is easy to see that $\mathbb{P}_{k+1} = \bigcup\{P(s, t) : s, t \text{ are finite } (k+1)\text{-anti-Sacks trees}\}$. We will show that every $P(s, t)$ is k-linked. Let $\{\langle s, F_i\rangle : i < k\} \subseteq P(s, t)$ and let $F = \bigcup_{i<k} F_i$. We must show that $\langle s, F\rangle \in \mathbb{P}_{k+1}$. The properties (a) and (c) are easily verified, so the only thing left to do is to show that $\lceil F{\restriction}\Delta_F \rceil$ is a $(k+1)$-anti-Sacks tree. Let $\sigma \in \lceil F{\restriction}\Delta_F \rceil$. If $|\sigma| < \mathsf{ht}(t)$, then, because $F{\restriction}\mathsf{ht}(t) = t$, it is possible to find an $i \in k$ such that $\sigma^\frown\langle i\rangle \notin \lceil F{\restriction}\Delta_F + 1 \rceil$. If $|\sigma| \geq \mathsf{ht}(t)$, then, for every $i < k$, σ only has (at most) one immediate successor in F_i and therefore it is always possible to find a $j \in k+1$ such that $\sigma^\frown\langle j\rangle \notin \lceil F{\restriction}\Delta_F + 1 \rceil$. $\qquad\square$

From these last two propositions we get the following result.

Corollary 4.3 *For each $k > 1$, $\mathfrak{m}_k \leq \mathsf{non}(\mathbf{AS}_{k+1})$ and $\mathfrak{m}_{2^k-1} \leq \mathsf{non}(\mathbf{SP}_k)$.*

Proof This follows easily from the proof of the Proposition 4.1 and the last proposition. $\qquad\square$

For the proof of the main theorem we will need the following notion.

Definition 4.4 *Given a regular cardinal κ and $\mathcal{I} \in \{\mathbf{SP}_n, \mathbf{AS}_k : n > 0, k > 1\}$, we will say that a set L is $\langle\kappa, \mathcal{I}\rangle$-Luzin if $|L| = \kappa$ and $\mathcal{I}{\restriction}L = [L]^{<\kappa}$.*

Observe that the existence of a $\langle\kappa, \mathcal{I}\rangle$-Luzin set implies that $\mathsf{non}(\mathcal{I}) \leq \kappa$. Recall that Cohen reals are added at every limit step of countable cofinality of a finite support iteration of arbitrary length. One common application of Cohen reals is that they are used to construct Luzin sets with special properties. The following lemma is one of those applications.

Lemma 4.5 *Let κ be a regular cardinal, let $i > 2$, $k > 1$ and let $\mathbb{L} = \langle\mathbb{L}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha \in \kappa\rangle$ be a finite support iteration of length κ such that $\mathbb{L}_\alpha \Vdash \text{``}\dot{\mathbb{Q}}_\alpha = \mathbb{P}_i * \mathbf{P}_k\text{''}$, then $\mathbb{L} \Vdash \text{``}$There is a $\langle\kappa, \mathbf{AS}_i\rangle$-Luzin set and there is a $\langle\kappa, \mathbf{SP}_k\rangle$-Luzin set$\text{''}$.*

Proof Working in $V[G]$, let $L = \{f_\alpha : \alpha \in \kappa \land \alpha$ has countable cofinality$\}$ be a family of Cohen reals such that each f_α is added at the α-th stage of the iteration. Using the Proposition 4.1, it is easy to show that $V[G] \models [L]^{<\kappa} \subseteq \mathbf{AS}_i \upharpoonright L$. On the other hand, if $T \in V[G]$ is such that $V[G] \models T$ is an i-anti-Sacks tree, then, by a standard reflection argument, there is an intermediate model such that $T \in V[G(\beta)]$. As a consequence, $V[G] \models \forall \gamma > \beta(f_\gamma \notin [T])$. This implies that $V[G] \models \mathbf{AS}_i \upharpoonright L \subseteq [L]^{<\kappa}$. The $\langle \kappa, \mathbf{SP}_k \rangle$-Luzin set is found in a similar way. $\square$

In the lemma above, it is clear that if we replace $\mathbb{L}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha = \mathbb{P}_i * \mathbf{P}_k$" by $\mathbb{L}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha = \mathbb{P}_i$", then, in the extension, we still have a $\langle \kappa, \mathbf{AS}_i \rangle$-Luzin set. The following theorem is the main tool we will use to prove the main result of this section.

Theorem 4.6 *If* ZFC *is consistent, then* ZFC $+ \forall i > 2$ $(\exists L_i$ $(L_i$ *is* $\langle \aleph_i, \mathbf{AS}_i \rangle$-*Luzin*$)) + \forall k > 1$ $(\exists L'_k$ $(L'_k$ *is* $\langle \aleph_{2^k}, \mathbf{SP}_k \rangle$-*Luzin*$))$ *is consistent.*

Proof Let $\mathbb{L} = \langle \mathbb{L}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha \in \omega_\omega \rangle$ be a finite support iteration of length ω_ω such that, for each $i > 1$ and each $\alpha \in [\omega_i, \omega_{i+1})$, $\mathbb{L}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha = \mathbb{P}_{i+1} * \mathbf{Q}_{i+1}$", where $\mathbf{Q}_{i+1} = \mathbf{P}_{i+1}$ when $i + 1$ is a number of the form $2^k + 1$ and $\mathbf{Q}_{i+1} = \{\emptyset\}$ in all the other cases (for $\alpha < \omega_2$, $\mathbb{L}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha = \{\emptyset\}$"). We will show that the extension is the model we are looking for. As usual in this work, we will only show that there are $\langle \aleph_i, \mathbf{AS}_i \rangle$-Luzin sets for every $i > 2$ (the rest can be done in a similar way). Using the lemma above, for each $i > 2$, in $V[G_{\omega_i}]$ there is a $\langle \aleph_i, \mathbf{AS}_i \rangle$-Luzin set L_i. The only thing left to do is to show that L_i remains $\langle \aleph_i, \mathbf{AS}_i \rangle$-Luzin in $V[G]$. Using that $\mathbb{L}$ is c.c.c. it is easy to see that, in $V[G]$, $[L_i]^{<\omega_i} \subseteq \mathbf{AS}_i \upharpoonright L_i$, so we only need to show that $\mathbf{AS}_i \upharpoonright L_i \subseteq [L_i]^{<\omega_i}$ holds in $V[G]$. First, using Lemma 3.4, Lemma 3.6 and Proposition 4.2, we observe that $\mathbb{L}_{[\omega_i, \omega_\omega]}$ strongly preserves non$(\mathbf{AS}_i)$ in L_i, so if $\dot{T}$ is a $\mathbb{L}_{[\omega_i, \omega_\omega]}$-name of a i-anti-Sacks tree, then, in $V[G_{\omega_i}]$, there is a $X \in \mathbf{AS}_i \upharpoonright L_i$ such that $\mathbb{L}_{[\omega_i, \omega_\omega]} \Vdash$ "$[\dot{T}] \cap L_i \subseteq X$". It follows that $\mathbf{AS}_i \upharpoonright L_i \subseteq [L_i]^{<\omega_i}$ holds in $V[G]$. $\square$

The actual value of $\mathfrak{c}$ in the model above may depend on V. For example, if $V \models$ GCH, then it is easy to see that $V[G] \models \mathfrak{c} = \aleph_{\omega+1}$.

Lemma 4.7 *Let κ be a regular cardinal, let $\mathcal{I} \in \{\mathbf{SP}_n, \mathbf{AS}_k : n > 0, k > 1\}$ and let L be a $\langle \kappa, \mathcal{I} \rangle$-Luzin set. If $\mathbb{P}$ is a forcing notion such that $|\mathbb{P}| < \kappa$, then $\mathbb{P}$ strongly preserves* non$(\mathcal{I})$ *in L.*

Proof We will do the case when $\mathcal{I} \in \{\mathbf{AS}_i : i > 1\}$; the other cases are similar. Let $\dot{A}$ be a $\mathbb{P}$-name of an i-anti-Sacks tree and let $\mathbb{P} = \{p_\alpha : \alpha \in \mu\}$. For each $\alpha \in \mu$

define $T_\alpha = \{s \in {}^{<\omega}k : p_\alpha \Vdash "s \in \dot{A}"\}$. It follows that each T_α defines an i-anti-Sacks tree. If $Y = \bigcup\{[T_\alpha] \cap L : \alpha \in \mu\}$ then $Y \in \mathbf{AS}_k$. If $x \in L$ and $p_\alpha \Vdash "x \in [\dot{A}]"$, then $x \in [T_\alpha] \cap L \subseteq Y$. $\square$

We are ready to prove the main result of this section.

Theorem 4.8 *If* ZFC *is consistent, then* ZFC $+ \forall k > 1\,(\mathfrak{m}_k = \mathsf{non}(\mathbf{AS}_{k+1}) = \aleph_{k+1}) + \forall i > 1\,(\mathfrak{m}_{2^i-1} = \mathsf{non}(\mathbf{SP}_i) = \aleph_{2^i}) + \mathsf{non}(\mathbf{SP}) = \aleph_{\omega+1}$ *is consistent.*

Proof Start with a model V like the one constructed in Theorem 4.6 and $V \models \mathfrak{c} = \aleph_{\omega+1}$. Using a standard bookkeeping argument, it is possible to construct a finite support iteration $\mathbb{P}$ of length $\omega_{\omega+1}$ of σ-k-linked forcings of size smaller than $\aleph_{k+1}$ (for every $k > 1$), such that any partial order which appears in an intermediate model is listed cofinally along the iteration. Now, using Lemmas 3.4, 3.6 and 4.7, it is possible to show that, for every $k > 2$, $\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{AS}_k)$ in L_k. If $G \subseteq \mathbb{P}$ is a generic filter over V, then $V[G] \models \mathsf{non}(\mathbf{AS}_k) \le \aleph_k$. We note that, as each small σ-k-linked forcing appears in an intermediate model in the iteration, we have $V[G] \models \aleph_{k+1} \le \mathfrak{m}_k$. As a consequence $V[G] \models \aleph_{k+1} = \mathfrak{m}_k = \mathsf{non}(\mathbf{AS}_{k+1})$. Using a similar argument, it is possible to show that, for each $i > 1$, $V[G] \models \aleph_{2^i} = \mathfrak{m}_{2^i-1} = \mathsf{non}(\mathbf{SP}_i)$. To finish the proof, use the fact that $\mathsf{non}(\mathbf{SP})$ does not have countable cofinality and that, for every $n \in \omega$, $\mathsf{non}(\mathbf{SP}_n) \le \mathsf{non}(\mathbf{SP})$ to show that $V[G] \models \mathsf{non}(\mathbf{SP}) = \mathfrak{c} = \aleph_{\omega+1}$. $\square$

It follows from $\mathbf{SP}_1 \subseteq \mathbf{SP}_2 \subseteq \mathbf{SP}_3 \subseteq \ldots$ that $\omega_1 = \mathsf{non}(\mathbf{SP}_1) \le \mathsf{non}(\mathbf{SP}_2) \le \mathsf{non}(\mathbf{SP}_3) \le \ldots \le \mathsf{non}(\mathbf{SP})$ and we proved in the theorem above that each inequality can be consistently strict. It is important to remark that none of these numbers is comparable with $\mathfrak{m}_{\sigma\text{-centered}}$. An argument for this can be found in Hrušák and Zindulka [6].

5 The covering number

It follows from the fact that $\mathbf{AS}_2 \subseteq \mathbf{AS}_3 \subseteq \ldots$ that $\mathfrak{c} = \mathsf{cov}(\mathbf{AS}_2) \ge \mathsf{cov}(\mathbf{AS}_3) \ge \ldots$. We can show that every pair of these numbers can be consistently different.

Proposition 5.1 *Let* $k > 1$, *if* ZFC *is consistent, then* ZFC $+ \mathsf{cov}(\mathbf{AS}_{k+1}) < \mathsf{cov}(\mathbf{AS}_k)$ *is consistent.*[3]

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[3]A similar theorem for the ideals $\mathbf{SP}_k$ can be proved using the same argument.

Proof Let V be a model such that $V \models \mathrm{cov}(\mathbf{AS}_k) = \mathfrak{c} = \omega_2$. Let $\mathbb{P}$ be a finite support iteration of length ω_1 of the $\mathbb{P}_{k+1}$ forcing defined above and let $G \subseteq \mathbb{P}$ be a generic filter over V. It follows that $\mathbb{P}$ is an iteration of σ-k-linked forcing notions and therefore $\mathbb{P}$ strongly preserves $\mathrm{non}(\mathbf{AS}_k)$. In $V[G]$, consider the family $C = \{V[G_\alpha] \cap {}^\omega(k+1) : \alpha < \omega_1\}$. Using Proposition 4.1, it is easy to show that $V[G] \models C \subseteq \mathbf{AS}_{k+1}$ and $V[G] \models \bigcup C = {}^\omega(k+1)$. As a consequence we have that $V[G] \models \mathrm{cov}(\mathbf{AS}_{k+1}) = \omega_1$. On the other hand, if $\{\dot{T}_\alpha : \alpha \in \omega_1\}$ is a collection of $\mathbb{P}$-names for k-anti-Sacks trees, then we can use the fact that $\mathbb{P}$ strongly preserves $\mathrm{non}(\mathbf{AS}_k)$ to show that there is a collection $\{C_\alpha : \alpha \in \omega_1\} \subseteq \mathbf{AS}_k$ such that if $x \in {}^\omega k$ and $x \notin \bigcup\{C_\alpha : \alpha \in \omega_1\}$, then $\mathbb{P} \Vdash$ "$x \notin \bigcup_{\alpha \in \omega_1}[\dot{T}_\alpha]$". This, together with $V \models \mathrm{cov}(\mathbf{AS}_k) > \omega_1$, implies that $V[G] \models \mathrm{cov}(\mathbf{AS}_{k+1}) < \mathrm{cov}(\mathbf{AS}_k)$. $\qquad\square$

An alternative proof of this proposition follows from the results proven in Newelski and Rosłanowski [9]. If $k > 1$ then a tree $T \subseteq {}^{<\omega}\omega$ is a *k-tree* if every $s \in T$ has at most k immediate successors. A forcing notion $\mathbb{P}$ has the *k-localization property* if $\mathbb{P} \Vdash$ "$\forall f \in {}^\omega\omega \, (\exists T \in V \, (T \text{ is a } k\text{-tree and } f \in [T]))$". It is easy to see that if $\mathbb{P}$ has the k-localization property, then $\mathbb{P} \Vdash$ "$\bigcup(\mathbf{AS}_{k+1} \cap V) = {}^\omega(k+1)$". Let $\mathbb{S}_k = \{T \subseteq {}^{<\omega}k : \forall s \in T(\exists t \in T(\forall i \in k(s \sqsubseteq t \wedge t^\frown i \in T)))\}$ be the k-Sacks forcing ordered by inclusion. It turns out that $\mathbb{S}_k$ is forcing equivalent to $\mathrm{Borel}({}^\omega k)/\mathbf{AS}_k$ and that if $\mathbb{P}$ is the countable support iteration or the countable support product of length ω_2 of the forcing $\mathbb{S}_k$, then $\mathbb{P}$ has the k-localization property (see [9]). As a consequence, in the extension $\mathrm{cov}(\mathbf{AS}_{k+1}) = \omega_1$ and $\mathrm{cov}(\mathbf{AS}_k) = \omega_2$.

Obviously it is impossible to separate infinitely many of the $\mathrm{cov}(\mathbf{SP}_n)$ at the same time. This suggests the following:

Question 5.2 *How many of the* $\mathrm{cov}(\mathbf{SP}_n)$ *can be separated at the same time?*

We do not even know how to separate three of them. Another question we have is the following:

Question 5.3 *Is it possible to get the consistency of* ZFC + $\forall k \in \omega \, (\mathrm{cov}(\mathbf{SP}) < \mathrm{cov}(\mathbf{SP}_k))$?

We are also interested in the relationship between $\mathrm{non}(\mathbf{SP})$ and $\mathrm{cov}(\mathbf{SP})$. It follows from the fact that the Cohen forcing is σ-centered that, in the Cohen model, $\mathrm{non}(\mathbf{SP}) < \mathrm{cov}(\mathbf{SP})$. However, we do not know if it is possible to construct a model where $\mathrm{non}(\mathbf{SP}) > \mathrm{cov}(\mathbf{SP})$.

Question 5.4 *Is* $\mathrm{non}(\mathbf{SP}) \leq \mathrm{cov}(\mathbf{SP})$?

A related question, as to whether $\text{non}(\mathbf{AS}_n) \leq \text{cov}(\mathbf{AS}_n)$ was asked in [9]. Finally, we would like to discuss about the relation of the cardinal numbers of the ideals $\mathbf{SP}_k$ and $\mathbf{AS}_{2^k}$. In this work we showed that these ideals share a lot of properties, however we do not know if they share the same cardinal invariants. There is a connection between 2^k-anti-Sacks trees and k-porous sets given by the following argument. Let $\varphi_k : 2^k \to {}^k 2$ be a bijective function. Let $\psi_k : {}^\omega(2^k) \to {}^\omega 2$ defined by $\psi_k(x) = \varphi_k(x(0))^\frown \varphi_k(x(1))^\frown \ldots$. Clearly, if $\psi_n(A) \in \mathbf{SP}_n$, then $A \in \mathbf{AS}_{2^n}$. We do not know if this can be used to show a relation between the cardinal invariants of the ideals $\mathbf{SP}_k$ and $\mathbf{AS}_{2^k}$.

Question 5.5 *Is* $\text{non}(\mathbf{SP}_k) = \text{non}(\mathbf{AS}_{2^k})$? *Is* $\text{cov}(\mathbf{SP}_k) = \text{cov}(\mathbf{AS}_{2^k})$?

6 Acknowledgements

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