Journal of Logic & Analysis 4:11 (2012) 1–24
ISSN 1759-9008

Relative set theory: Strong stability

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Abstract: **GRIST** is an axiomatic framework for nonstandard set theory with many "levels of standardness." The paper establishes a number of general consequences of **GRIST**, in particular, a very strong form of Transfer principle.

2010 Mathematics Subject Classification 26E35 (primary); 03E70, 03H05 (secondary)

Keywords: nonstandard analysis, nonstandard set theory, IST, RIST, level, relative infinitesimal, transfer, S–continuity

This is the last in a series of three articles devoted to **GRIST**, an axiomatic presentation of nonstandard analysis with many "levels of standardness." The two previous papers, [4] and [5], will be referred to as RST and RST2, respectively.

It is shown in RST that **GRIST** is complete over **ZFC**: If an extension of **GRIST** proves a theorem that is not provable in **GRIST**, then it proves a theorem in the language of **ZFC** that is not provable in **ZFC** [see Proposition 6.5]. In other words, no additional principles can be added to **GRIST** while keeping it conservative over **ZFC**. Yet in mathematical applications it is sometimes awkward to argue directly from the axioms of **GRIST**. It is convenient to have at one's disposal other principles, provable in **GRIST**, but tailor–made for certain kinds of applications. A number of such consequences of **GRIST** is derived in RST, Section 12; see also RST2, Proposition 1.10. For applications of relative set theory see RST2 and [3, 6, 10].

This paper focuses on deducing some further useful principles in **GRIST**. Foremost among them is Strong Stability, perhaps the ultimate generalization of Transfer. Section 1 begins with a formulation of Strong Stability. Strong Stability is then used to prove that levels represented by elements of a given set are precisely those from a finite union of singletons and closed intervals (in the ordering of levels by inclusion).

Section 2 contains the proof of Strong Stability in **GRIST**. It relies heavily on the development of **GRIST** in RST. Counterexamples to some "natural" strengthenings of Strong Stability are also constructed there.

Section 3 deals with some variants of Idealization and Choice that are provable in **GRIST**. It also presents a generalization of Robinson's Lemma due to Andreev.

Published: June 2012 doi: 10.4115/jla.2012.4.11

In Section 4 we complete the study of $(\mathbf{V}_1, \mathbf{V}_2)$–continuity initiated in RST2.

In **GRIST**, the ordering of levels is dense (and there is a coarsest level). Alternatively, one can postulate that levels are discretely ordered (each level has an immediate successor, and each but the coarsest level has an immediate predecessor). There are other possibilities, and weaker theories (**FRIST** [2], Péraire's **RIST** [9]) agnostic on the details of the ordering of levels. In Section 5 we show that all these theories prove exactly the same *open* formulas (formulas where no quantification over levels occurs).

Finally, in Section 6 we fill a gap in the proof of Proposition 6.10 in RST, and add a few easy but useful observations on **GRIST** that have not been made explicit in RST.

For convenient reference, the axioms of **GRIST** are listed below [see RST2, page 4]. In addition to **ZFC**, they are

Relativization *The conjunction of:*
(o) $(\forall \mathbf{U}, \mathbf{V})[(\forall x)(x \in \mathbf{U} \leftrightarrow x \in \mathbf{V}) \rightarrow \mathbf{U} = \mathbf{V}]$;
(i) $(\forall x)(\exists \mathbf{V})[x \in \mathbf{V} \ \wedge \ (\forall \mathbf{U})(x \in \mathbf{U} \rightarrow \mathbf{V} \subseteq \mathbf{U})]$
(this uniquely determined level $\mathbf{V}$ is denoted $\mathbf{V}(x)$; $\mathbf{V}(\cdot) := \mathbf{V}(0)$ is the coarsest level);
(ii) $(\forall \mathbf{V})[0 \in \mathbf{V} \ \wedge \ (\exists x \in \mathbf{V})(\forall \mathbf{U})(x \in \mathbf{U} \rightarrow \mathbf{V} \subseteq \mathbf{U})]$;
(iii) $(\forall \mathbf{U}, \mathbf{V})(\mathbf{U} \subseteq \mathbf{V} \ \vee \ \mathbf{V} \subseteq \mathbf{U})$;
(iv) $(\forall \mathbf{U})(\exists \mathbf{V})(\mathbf{U} \subset \mathbf{V})$;
(v) $(\forall \mathbf{U}, \mathbf{V})(\mathbf{U} \subset \mathbf{V} \rightarrow (\exists \mathbf{W})(\mathbf{U} \subset \mathbf{W} \subset \mathbf{V})$.

Transfer (*or* **Stability**)

For all $\mathbf{U} \subseteq \mathbf{V}$ *and all* $x_1, \ldots, x_k \in \mathbf{U}$, $\quad \mathcal{P}(x_1, \ldots, x_k; \mathbf{U}) \leftrightarrow \mathcal{P}(x_1, \ldots, x_k; \mathbf{V})$.

Standardization

For all $\mathbf{U}$ *and all* $A, x_1, \ldots, x_k$, *either* $(\forall \mathbf{V})(\mathbf{U} \subseteq \mathbf{V})$ *or there exist* $\mathbf{V} \subset \mathbf{U}$ *and* $B \in \mathbf{V}$ *such that, for every* $\mathbf{W}$ *with* $\mathbf{V} \subseteq \mathbf{W} \subset \mathbf{U}$,

$$(\forall y \in \mathbf{W})(y \in B \leftrightarrow y \in A \ \wedge \ \mathcal{P}(y, x_1, \ldots, x_k; \mathbf{W})).$$

Idealization

For all $\mathbf{U}, \mathbf{V}, A$ *such that* $A \in \mathbf{U} \subset \mathbf{V}$, *and all* $x_1, \ldots, x_k$,

$(\forall a \in \mathcal{P}^{\mathbf{fin}} A)(\forall \mathbf{W} \subset \mathbf{V})\ [a \in \mathbf{W} \rightarrow (\exists y)(\forall x \in a)\mathcal{P}(x, y, x_1, \ldots, x_k; \mathbf{V})] \ \leftrightarrow$

$(\exists y)(\forall x \in A)(\forall \mathbf{W} \subset \mathbf{V})[x \in \mathbf{W} \rightarrow \mathcal{P}(x, y, x_1, \ldots, x_k; \mathbf{V})]$.

Granularity

For all $x_1, \ldots, x_k$, *if* $(\exists \mathbf{U})\mathcal{P}(x_1, \ldots, x_k; \mathbf{U})$, *then*

$$(\exists \mathbf{U})[\mathcal{P}(x_1, \ldots, x_k; \mathbf{U}) \ \wedge \ (\forall \mathbf{V})(\mathbf{V} \subset \mathbf{U} \rightarrow \neg\, \mathcal{P}(x_1, \ldots, x_k; \mathbf{V}))].$$

1 Strong Stability and 'sets of levels'.

The key principle of **GRIST** is **Transfer**, also referred to as **Stability**:

For all $\mathbf{V} \subseteq \mathbf{V}'$ *and all* $x_1, \ldots, x_k \in \mathbf{V}$,

$$\mathcal{P}(x_1, \ldots, x_k; \mathbf{V}) \leftrightarrow \mathcal{P}(x_1, \ldots, x_k; \mathbf{V}'),$$

where $\mathcal{P}(x_1, \ldots, x_k; \mathbf{V})$ is any $\mathbf{V}$-formula, ie, a formula where all quantifiers over levels are of the form $(\forall \mathbf{W} \supseteq \mathbf{V})$ or $(\exists \mathbf{W} \supseteq \mathbf{V})$.

The most irksome limitation of Transfer is the restriction $x_1, \ldots, x_k \in \mathbf{V}$; but without it the principle fails as stated [consider $\mathbf{V} \subset \mathbf{V}'$, $x \in \mathbf{V}' \smallsetminus \mathbf{V}$, and the formula $\mathcal{P}(x; \mathbf{V})$: $x \in \mathbf{V}$]. The first step towards transgressing this limitation is made in the Local Transfer principle.

Local Transfer [RST2, Proposition 1.10 (6)]: *For any sets* $x_{k+1}, \ldots, x_n$ *and any* $\mathbf{V}_0$ *there is* $\mathbf{V}' \supset \mathbf{V}_0$ *such that, for all* $\mathbf{V}_0 \subseteq \mathbf{V} \subset \mathbf{V}'$ *and all* $x_1, \ldots, x_k \in \mathbf{V}_0$,

$$\mathcal{P}(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n; \mathbf{V}_0) \leftrightarrow \mathcal{P}(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n; \mathbf{V}).$$

We also have

Support Principle [RST2, Proposition 1.10 (5)]: *Given a* $\mathbf{V}$*-formula* $\mathcal{P}(x_1, \ldots, x_k; \mathbf{V})$ *and sets* $x_1, \ldots, x_k$, *there is a finite set* $\{v_0, v_1, \ldots, v_n\}$ *such that* $\mathbf{V}(\cdot) = \mathbf{V}(v_0) \subset \mathbf{V}(v_1) \subset \ldots \subset \mathbf{V}(v_n)$ *and for all* $i \leq n$ *and all* $\mathbf{V}$ *with* $\mathbf{V}(v_i) \subseteq \mathbf{V} \subset \mathbf{V}(v_{i+1})$ [$\mathbf{V}(v_i) \subseteq \mathbf{V}$ *if* $i = n$],

$$\mathcal{P}(x_1, \ldots, x_k; \mathbf{V}(v_i)) \leftrightarrow \mathcal{P}(x_1, \ldots, x_k; \mathbf{V}) \leftrightarrow \neg\, \mathcal{P}(x_1, \ldots, x_k; \mathbf{V}(v_{i+1})).$$

In this section we formulate a principle (Strong Stability) that generalizes all of the above, and give an example that illustrates its use.

Definition 1.1 (RST, Definition 8.3; see also RST, Proposition 8.6, and Section 2 of this paper.) A set L is a **level set** if for all $x, y \in L$, $\mathbf{V}(x) = \mathbf{V}(y)$ implies $x = y$.

Level sets are finite, and the relation $\sqsubseteq$ defined on L by $x \sqsubseteq y \leftrightarrow \mathbf{V}(x) \subseteq \mathbf{V}(y)$ is a well-ordering. We always describe level sets in the increasing order by $\sqsubseteq$; ie, if $L = \{z_0, z_1, \ldots, z_\ell\}$ is a level set, then $\mathbf{V}(z_0) \subset \mathbf{V}(z_1) \subset \ldots \subset \mathbf{V}(z_\ell)$.

Definition 1.2 Let L be a level set. We write $\mathbf{V} \cong_L \mathbf{V}'$ if $\mathbf{V} \subseteq \mathbf{V}(z) \leftrightarrow \mathbf{V}' \subseteq \mathbf{V}(z)$ and $\mathbf{V}(z) \subseteq \mathbf{V} \leftrightarrow \mathbf{V}(z) \subseteq \mathbf{V}'$ hold for all $z \in L$.

In other words, if $L = \{z_0, z_1, \ldots, z_\ell\}$, then $\mathbf{V} \cong_L \mathbf{V}'$ means that either $\mathbf{V} = \mathbf{V}' = \mathbf{V}(z_j)$ for some $j \leq \ell$, or $\mathbf{V}, \mathbf{V}' \subset \mathbf{V}(z_0)$, or $\mathbf{V}(z_j) \subset \mathbf{V}, \mathbf{V}' \subset \mathbf{V}(z_{j+1})$ for some $j < \ell$, or $\mathbf{V}(z_\ell) \subset \mathbf{V}, \mathbf{V}'$. Thus $\cong_L$ classifies all levels into $2\ell + 3$ classes.

Recall [RST2, page 4] that $\mathcal{P}(x_1, \dots, x_k, y_1, \dots, y_\ell; \mathbf{V}, \mathbf{V}_1, \dots, \mathbf{V}_n)$ denotes a formula of the language of **GRIST** where all quantifiers over levels are of the form $(\forall \mathbf{W} \supseteq \mathbf{V})$ or $(\exists \mathbf{W} \supseteq \mathbf{V})$. For brevity, we often write $\bar{x}$ and $\bar{y}$ for the sequences $x_1, \dots, x_k$ and $y_1, \dots, y_\ell$, respectively. Then $\langle \bar{x} \rangle$ denotes $\langle x_1, \dots, x_k \rangle$, etc.

STRONG STABILITY: *Given $y_1, \dots, y_\ell$, there is a level set L such that if $\mathbf{V} \subset \mathbf{V}_1$, $\mathbf{V} \subset \mathbf{V}'_1$ and $\mathbf{V}_1 \cong_L \mathbf{V}'_1$, then*

$$(\forall \bar{x} \in \mathbf{V})(\mathcal{P}(\bar{x}, \bar{y}; \mathbf{V}, \mathbf{V}_1) \leftrightarrow \mathcal{P}(\bar{x}, \bar{y}; \mathbf{V}, \mathbf{V}'_1)).$$

The proof is postponed until Section 2, where also a stronger version, for several levels simultaneously, can be found. Here we give a simple example of an application of this principle. We show that every "set of levels" is a finite union of singletons and closed intervals in the linear ordering of levels by $\subseteq$.

Theorem 1.3 *For every $X \neq \varnothing$ there exist level sets $L = \{\alpha_0, \dots, \alpha_n\}$ and $L' = \{\alpha'_0, \dots, \alpha'_n\}$ such that $\mathbf{V}(\alpha_i) \subseteq \mathbf{V}(\alpha'_i)$ for all $i \leq n$, $\mathbf{V}(\alpha'_i) \subset \mathbf{V}(\alpha_{i+1})$ for all $i < n$, and*

(1) $\quad (\forall x \in X)(\exists i \leq n)(\mathbf{V}(\alpha_i) \subseteq \mathbf{V}(x) \subseteq \mathbf{V}(\alpha'_i))$,

(2) $\quad (\forall \mathbf{V})(\forall i \leq n)[\mathbf{V}(\alpha_i) \subseteq \mathbf{V} \subseteq \mathbf{V}(\alpha'_i) \to (\exists x \in X)(\mathbf{V}(x) = \mathbf{V})]$

if X is finite. If X is infinite, (1) and (2) hold with $\mathbf{V}(\alpha_n) \subseteq \mathbf{V}(x) \subseteq \mathbf{V}(\alpha'_n)$ and $\mathbf{V}(\alpha_n) \subseteq \mathbf{V} \subseteq \mathbf{V}(\alpha'_n)$ replaced by $\mathbf{V}(\alpha_n) \subseteq \mathbf{V}(x)$ and $\mathbf{V}(\alpha_n) \subseteq \mathbf{V}$, respectively.

Lemma 1.4 *If $\mathbf{V}(\alpha) \subset \mathbf{V}(\alpha')$ and $(\exists x \in X)(\mathbf{V}(x) = \mathbf{V})$ holds for all $\mathbf{V}$ such that $\mathbf{V}(\alpha) \subset \mathbf{V} \subset \mathbf{V}(\alpha')$, then also*

(A) $\quad (\exists x \in X)(\mathbf{V}(x) = \mathbf{V}(\alpha))$ *and*

(B) $\quad (\exists x \in X)(\mathbf{V}(x) = \mathbf{V}(\alpha'))$.

Proof (B) Let a be a finite set, $\mathbf{V}(a) \subset \mathbf{V}(\alpha')$. Fix $\mathbf{V}$ such that $\mathbf{V}(a), \mathbf{V}(\alpha) \subset \mathbf{V} \subset \mathbf{V}(\alpha')$ and $x \in X$ such that $\mathbf{V}(x) = \mathbf{V}$. By RST2, Proposition 1.10 (13), $y \in a$ implies $y \in \mathbf{V}(a)$, so $y \neq x$. Hence $(\forall^{\mathbf{fin}} a)[\mathbf{V}(a) \subset \mathbf{V}(\alpha') \to (\exists x \in \mathbf{V}(\alpha'))(\forall y \in a)(x \in X \wedge y \neq x)]$. By **GRIST** Idealization we obtain $x \in X$, $x \in \mathbf{V}(\alpha')$, such that $y \neq x$ holds for all $y \in X$ with $\mathbf{V}(y) \subset \mathbf{V}(\alpha')$. Then $x \in X$ and $\mathbf{V}(x) = \mathbf{V}(\alpha')$.

(A) If $\mathbf{V}(\alpha) = \mathbf{V}(\cdot)$ [the coarsest level], consider the statement

$$\mathcal{P}(X; \mathbf{V}) : \quad (\exists x \in X)(x \in \mathbf{V}).$$

$\mathcal{P}(X;\mathbf{V})$ holds for all $\mathbf{V}$ such that $\mathbf{V}(\cdot) \subset \mathbf{V} \subset \mathbf{V}(\alpha')$, by the assumption. By Granularity, there is a coarsest level $\mathbf{V}$ for which $\mathcal{P}(X;\mathbf{V})$ holds. From density of levels [Relativization (v)] we conclude that $\mathcal{P}(X;\mathbf{V}(\cdot))$: $(\exists x \in X)(x \in \mathbf{V}(\cdot))$ holds.

If $\mathbf{V}(\cdot) \subset \mathbf{V}(\alpha)$ and for every $\mathbf{V} \subset \mathbf{V}(\alpha)$ there is $x \in X$ such that $\mathbf{V} \subseteq \mathbf{V}(x) \subset \mathbf{V}(\alpha)$, the argument in the proof of (B) [with $\mathbf{V}(\cdot)$ in place of $\mathbf{V}(\alpha)$ and $\mathbf{V}(\alpha)$ in place of $\mathbf{V}(\alpha')$] shows that $(\exists x \in X)(\mathbf{V}(x) = \mathbf{V}(\alpha))$.

It remains to consider the case when there is a $\mathbf{V} \subset \mathbf{V}(\alpha)$ such that $\neg(\exists x \in X)(\mathbf{V} \subseteq \mathbf{V}(x) \subset \mathbf{V}(\alpha))$. Fix such $\mathbf{V} =: \overline{\mathbf{V}}$. For every finite a with $\mathbf{V}(a) \subset \overline{\mathbf{V}}$ there is $Y \in \overline{\mathbf{V}}$, Y finite, such that $(\forall x \in a)(x \in X \to x \in Y)$ [let $Y := a$]. By **GRIST** Idealization, there is a finite $Y \in \overline{\mathbf{V}}$ such that $(\forall x \in X)(\mathbf{V}(x) \subset \overline{\mathbf{V}} \to x \in Y)$. Let $Z := X \smallsetminus Y$ and note that $(\forall x)(x \in Z \leftrightarrow x \in X \wedge \mathbf{V}(\alpha) \subseteq \mathbf{V}(x))$ [$Y \in \overline{\mathbf{V}}$ is finite, so $x \in Y \to x \in \overline{\mathbf{V}}$, by RST2, Proposition 1.10 (13)]. As in the proof of the $\mathbf{V}(\alpha) = \mathbf{V}(\cdot)$ case, we consider

$$\mathcal{P}(Z;\mathbf{V}) : \quad (\exists x \in Z)(x \in \mathbf{V}).$$

We observe that $\mathcal{P}(Z;\mathbf{V}) \leftrightarrow \mathcal{P}(X;\mathbf{V})$ for $\mathbf{V} \supseteq \mathbf{V}(\alpha)$, and hence $\mathcal{P}(Z;\mathbf{V})$ holds for all $\mathbf{V}$ such that $\mathbf{V}(\alpha) \subset \mathbf{V} \subset \mathbf{V}(\alpha')$. Also, $\mathcal{P}(Z;\mathbf{V})$ fails for $\mathbf{V} \subset \mathbf{V}(\alpha)$. By Granularity, $\mathcal{P}(Z;\mathbf{V}(\alpha))$ holds. Hence $(\exists x \in Z)(x \in \mathbf{V}(\alpha))$; as $x \in Z \to \mathbf{V}(x) \supseteq \mathbf{V}(\alpha)$, we have $x \in Z \subseteq X$ and $\mathbf{V}(x) = \mathbf{V}(\alpha)$. □

Proof of Theorem 1.3 We apply Strong Stability to the statement

$$\mathcal{P}(X;\mathbf{V}(\cdot),\mathbf{V}) : \quad (\exists x \in X)(\mathbf{V}(x) = \mathbf{V})$$

[in detail: $(\exists x \in X)(\forall \mathbf{W} \supseteq \mathbf{V}(\cdot))(x \in \mathbf{W} \leftrightarrow \mathbf{V} \subseteq \mathbf{W})$] and obtain a level set $M = \{\gamma_0,\dots,\gamma_k\}$ such that, wlog, $\mathbf{V}(\gamma_0) = \mathbf{V}(\cdot)$ and for all $i \le k$, $\mathbf{V}(\gamma_i) \subset \mathbf{V},\mathbf{V}' \subset \mathbf{V}(\gamma_{i+1})$ [$\mathbf{V}(\gamma_i) \subset \mathbf{V},\mathbf{V}'$ if $i = k$] implies $\mathcal{P}(X;\mathbf{V}(\cdot),\mathbf{V}) \leftrightarrow \mathcal{P}(X;\mathbf{V}(\cdot),\mathbf{V}')$. By Lemma 1.4, if $\mathcal{P}(X;\mathbf{V}(\cdot),\mathbf{V})$ holds for all $\mathbf{V}(\gamma_i) \subset \mathbf{V} \subset \mathbf{V}(\gamma_{i+1})$, then also $\mathcal{P}(X;\mathbf{V}(\cdot),\mathbf{V}(\gamma_i))$ and $\mathcal{P}(X;\mathbf{V}(\cdot),\mathbf{V}(\gamma_{i+1}))$ hold. By Proposition 6.4 proved in Section 6 and Examples (2) and (3) that precede it, the sets $\{z \in M : (\exists x \in X)(\mathbf{V}(x) = \mathbf{V}(z))\}$ and $\{z \in M : (\forall \mathbf{V})[\mathbf{V} \supset \mathbf{V}(z) \to (\exists x \in X)(\mathbf{V}(z) \subset \mathbf{V}(x) \subseteq \mathbf{V})]\}$ exist. From these sets one easily obtains the sets L and L' as in the Theorem, by amalgamating adjacent intervals when necessary. □

By RST2, Proposition 2.15, the level sets L and L' can be taken to be sets of natural numbers: $L = \{k_0,\dots,k_n\}$ and $L' = \{k'_0,\dots,k'_n\}$, with $\mathbf{V}(k_i) \subseteq \mathbf{V}(k'_i)$ for all $i \le n$ and $\mathbf{V}(k'_i) \subset \mathbf{V}(k_{i+1})$ for all $i < n$. We can assume that $\mathbf{V}(k_i) = \mathbf{V}(k'_i)$ implies $k_i = k'_i$. [$\{k_i : \mathbf{V}(k_i) = \mathbf{V}(k'_i)\}$ is a set, again by Proposition 6.4, because the formula $(\exists i \le n)(z = k_i \wedge z \sqsubseteq k'_i \wedge k'_i \sqsubseteq z)$ is stable in z.]

Let $[k,k'] := \{i \in \mathbb{N} : k \leq i \leq k'\}$ and $[k,\infty) := \{i \in \mathbb{N} : k \leq i\}$. We define $S' := \bigcup_{i<n}[k_i,k_i']$ and $S := S' \cup [k_n,k_n']$ if X is finite; $S := S' \cup [k_n,\infty)$ if X is infinite. Then

$$(\forall x \in X)(\exists i \in S)(\mathbf{V}(x) = \mathbf{V}(i)) \text{ and } (\forall i \in S)(\exists x \in X)(\mathbf{V}(x) = \mathbf{V}(i)).$$

In the terminology introduced on page 12, every set is level–equivalent to a finite union of singletons and closed intervals on $\mathbb{N}$.

2 Proof of Strong Stability in GRIST.

This section relies heavily on the development of **GRIST** as given in RST. For this reason, it is convenient here to work with the original formulation of **GRIST** in terms of $\sqsubseteq$, rather than use the language of levels (**GRIST**$^\heartsuit$) introduced in RST2.

In the $\in$–$\sqsubseteq$–language, L is a **level set** if for all $x,y \in L$, $x \boxminus y$ implies $x = y$. We recall that $x \boxminus y$ is shorthand for $x \sqsubseteq y \wedge y \sqsubseteq x$.

We write $\alpha \cong_L \beta$ if $\alpha \sqsubseteq \gamma \leftrightarrow \beta \sqsubseteq \gamma$ and $\gamma \sqsubseteq \alpha \leftrightarrow \gamma \sqsubseteq \beta$ hold for all $\gamma \in L$.

In other words, if $L = \{\gamma_0, \gamma_1, \ldots, \gamma_\ell\}$, then $\alpha \cong_L \beta$ means that either $\alpha \boxminus \beta \boxminus \gamma_j$ for some $j \leq \ell$, or $\alpha, \beta \sqsubset \gamma_0$, or $\gamma_j \sqsubset \alpha, \beta \sqsubset \gamma_{j+1}$ for some $j < \ell$, or $\gamma_\ell \sqsubset \alpha, \beta$.

$\mathcal{P}(x_1,\ldots,x_k,y_1,\ldots,y_\ell;z_1,\ldots,z_n)$ denotes a formula of the $\in$-$\sqsubseteq$-language where the variables $z_1,\ldots,z_n$ appear only in the scope of $\sqsubseteq$ [RST, Definition 12.25].

The Strong Stability principle in this language goes as follows [we recall that $\mathbb{S}_\alpha := \{x : x \sqsubseteq \alpha\}$, and $\mathcal{P}^\alpha$ is the formula obtained from $\mathcal{P}$ by replacing every occurrence of $\sqsubseteq$ with $\sqsubseteq_\alpha$, defined by $x \sqsubseteq_\alpha y \leftrightarrow x \sqsubseteq y \ \vee \ x \sqsubseteq \alpha$].

Theorem 2.1 (**Strong Stability**) *Given $y_1,\ldots,y_\ell$, there is a level set L, independent of $\mathcal{P}$, such that if $\alpha \sqsubset \beta$, $\alpha \sqsubset \beta'$ and $\beta \cong_L \beta'$, then*

$$(\forall \bar{x} \in \mathbb{S}_\alpha)(\mathcal{P}^\alpha(\bar{x},\bar{y};\beta) \leftrightarrow \mathcal{P}^\alpha(\bar{x},\bar{y};\beta')).$$

One can take $L = \operatorname{ran}\vec{u}$ where $\vec{u}$ is a pedigree for $\langle\bar{y}\rangle$ over some $A \in \mathbb{S}_0$. The notion of pedigree is the key technical tool for detailed study of **GRIST**. Roughly speaking, types of objects in the universe of **GRIST** are described by stratified ultrafilters. Pedigrees are finite sequences of stratified ultrafilters that describe, level by level, how the object realizes its type. We state the definitions of these concepts here for convenience [see RST, Sections 9 and 10].

We recall that βX is the set of all ultrafilters over X (the Stone-Čech space over X), and $U \sim V$ means that $U \cap V$ is an ultrafilter. For an arbitrary nonempty set A we define by recursion on ordinals:

(0) $\boldsymbol{\beta}_0 A := A$.

(1) For $\xi > 0$, $\boldsymbol{\beta}_{<\xi} A := \bigcup_{\eta<\xi} \boldsymbol{\beta}_\eta A$ and $\boldsymbol{\beta}_\xi A := \boldsymbol{\beta}_{<\xi} A \cup \{U \in \beta(\boldsymbol{\beta}_{<\xi} A) : U$ is nonprincipal and $\boldsymbol{\beta}_{<\eta} A \notin U$ for any $\eta < \xi\} = \boldsymbol{\beta}_{<\xi} A \cup \{U \in \beta(\boldsymbol{\beta}_{<\xi} A) : U$ is nonprincipal and there is no $V \in \boldsymbol{\beta}_{<\xi} A$ such that $U \sim V\}$.

Elements of $\boldsymbol{\beta}_\infty A := \bigcup_{\xi \in \mathbb{O}n} \boldsymbol{\beta}_\xi A$ are called **stratified ultrafilters over** A. For $U \in \boldsymbol{\beta}_\infty A$ we let $\operatorname{Dom} U := A$. As usual, the recursive definition assigns to each stratified ultrafilter an ordinal **rank**. Stratified ultrafilters of rank 1 are the nonprincipal ultrafilters over A; stratified ultrafilters of rank 2 are the nonprincipal ultrafilters over βA that concentrate on nonprincipal ultrafilters over A [ie, such that $(\beta A \setminus A) \in U$], and so on.

Let $x \in A \in \mathbb{S}_\alpha$. An α-**pedigree for** x **over** A is a sequence $\vec{u} = \langle u_n : n \leq \nu \rangle$ where $\nu \in \omega$ and
(i) every u_n is a stratified ultrafilter over A [ie, $u_n \in \boldsymbol{\beta}_\infty A$];
(ii) $u_0 \sqsubseteq \alpha$; $u_\nu = x$;
(iii) $\alpha \sqsubset u_1 \wedge (\forall n, m)(1 \leq n < m \leq \nu \to u_n \sqsubset u_m)$;
(iv) $(\forall z \sqsubset u_{n+1})(z \in u_n \to u_{n+1} \in z)$, for all $n < \nu$.

The ultrafilter u_0 is called the α-**type of** x **over** A and denoted $\mathbf{tp}_\alpha(x; A)$. We also use $\vec{u}^+ := \langle u_n : 0 < n \leq \nu \rangle$. We write $x \mathbf{M}_\alpha U$ as shorthand for: "There exists a [good; see Proposition 2.7] α-pedigree $\vec{u} = \langle u_n : n \leq \nu \rangle$ for x over some $A \in \mathbb{S}_\alpha$ such that $U = u_0$," and note that U is then an α-type of x. **Pedigree** and **type** mean 0-pedigree and 0-type, respectively.

We also recall that, for $a \in A$ and $U \in \boldsymbol{\beta}_\xi B$, $U_a \in \boldsymbol{\beta}_\xi (A \times B)$ is the unique stratified ultrafilter such that $\overline{\pi_1}(U_a) = a$ and $\overline{\pi_2}(U_a) = U$ [RST, Proposition 9.7].

The main technical result needed for the proof of Theorem 2.1 is the following proposition.

Proposition 2.2 *Let $\vec{u} = \langle u_0, u_1, \ldots, u_i, u_{i+1}, \ldots, u_\nu \rangle$ be a pedigree for $a \in A \in \mathbb{S}_0$ over A, and let $V, B \in \mathbb{S}_0$, $V \in \boldsymbol{\beta}_1 B \setminus \boldsymbol{\beta}_0 B$. Let $u_i \sqsubset \beta \sqsubset u_{i+1}$ [$u_\nu \sqsubset \beta$ if $i = \nu$], and let $b \boxminus \beta$ be such that $b \mathbf{M}_0 V$ [ie, b has the pedigree $\langle V, b \rangle$ over B; see RST, Proposition 12.15]. Then the pedigree $\vec{v}$ for $\langle b, a \rangle$ over $B \times A$ has the form*

$$\langle v_0, \ldots, v_m, (u_i)_b, (u_{i+1})_b, \ldots, (u_\nu)_b \rangle$$

where $v_m = u_i^{\bowtie} V$ is defined in Proposition 2.4 below, and $v_0, \ldots, v_m$ are independent of the choice of β and b.

Proposition 2.2 shows how to extend a pedigree for a set a to a pedigree that also "fixes" a particular level β. The proof requires two lemmas.

Lemma 2.3 *$\langle (u_i)_b, (u_{i+1})_b, \dots, (u_\nu)_b \rangle$ is the β-pedigree for $\langle b, a\rangle$ over $B \times A$.*

Proof Let $f : A \to B \times A$ be defined by $f(x) = \langle b, x\rangle$. By RST, Theorem 10.10 [see also RST, Proposition 10.14], the range of the β-pedigree for $\langle b, a\rangle$ is $\{\bar{f}(u_i), \bar{f}(u_{i+1}), \dots, \bar{f}(u_\nu)\}$. We prove that $\overline{\pi_1}(\bar{f}(u_j)) = b$ and $\overline{\pi_2}(\bar{f}(u_j)) = u_j$, for all $i \le j \le \nu$. We begin with noticing that $\pi_1(f(x)) = b$ and $\pi_2(f(x)) = x$. It is easily verified, from RST, Definition 9.3, that $\overline{k_b}(u) = b$ for all $u \in \boldsymbol{\beta}_\infty A$, where $k_b : A \to B$ is the constant function with value b. From this and RST, Proposition 9.4, we get $\overline{\pi_1}(\bar{f}(u_j)) = \overline{\pi_1 \circ f}(u_j) = \overline{k_b}(u_j) = b$ and $\overline{\pi_2}(\bar{f}(u_j)) = \overline{\pi_2 \circ f}(u_j) = \overline{Id_A}(u_j) = u_j$.

From the uniqueness in RST, Proposition 9.7, it follows that $\bar{f}(u_j) = (u_j)_b$. Finally, we notice that $(u_j)_b \boxminus_\beta u_j$: we have $(u_j)_b \sqsubseteq_\beta u_j$ because $(u_j)_b$ is $\in$-definable from u_j and $b \in \mathbb{S}_\beta$, and $u_j \sqsubseteq_\beta (u_j)_b$ because $u_j = \overline{\pi_2}((u_j)_b)$.

From these observations it follows that $\langle (u_i)_b, \dots, (u_\nu)_b\rangle$ is the β-pedigree for $\langle b, a\rangle$ over $B \times A$. $\square$

Proposition 2.4 **(ZFC)** *Let $V \in \boldsymbol{\beta}_1 B \smallsetminus \boldsymbol{\beta}_0 B$; for every $U \in \boldsymbol{\beta}_\xi A$ there is a unique $W \in \boldsymbol{\beta}_{\xi+1}(B \times A)$ such that $\{U_y : y \in B\} \in W$ and $\overline{\pi_1}(W) = V$, $\overline{\pi_2}(W) = U$. We denote this unique W by $U^{\bowtie}V$.*

Proof Assume $U \in \boldsymbol{\beta}_\xi A \smallsetminus \boldsymbol{\beta}_{<\xi} A$ and let $\tau_U : B \to \beta_\xi(B \times A)$ be defined by $\tau_U(y) = U_y$.

Existence: We let $W := \tau_U[V]$. As τ_U is one-one and $\operatorname{rank}(U_y) = \operatorname{rank} U$ for all $y \in B$, clearly $W \in \boldsymbol{\beta}_{\xi+1}(B \times A) \smallsetminus \boldsymbol{\beta}_\xi(B \times A)$. If $Y \in V$, then $\tau_U[Y] \in W$ and $Y = \overline{\pi_1}[\tau_U[Y]] \in \overline{\pi_1}[W]$. Both V and $\overline{\pi_1}[W]$ are ultrafilters over B, so $V = \overline{\pi_1}[W]$ and $V = \mathfrak{m}(V) = \mathfrak{m}(\overline{\pi_1}[W]) = \overline{\pi_1}(W)$.

$\overline{\pi_2}[W]$ is generated by sets of the form $\overline{\pi_2}[\tau_U[Y]]$ where $Y \in V$. But $\overline{\pi_2}[\tau_U[Y]] = \{\overline{\pi_2}(U_y) : y \in Y\} = \{U : y \in Y\} = \{U\}$. So $\overline{\pi_2}[W]$ is a principal ultrafilter generated by U and $\overline{\pi_2}(W) = \mathfrak{m}(\overline{\pi_2}[W]) = U$.

Uniqueness: Let W have the required properties. The map τ_U is one-one, so $\tau_U^{-1}(U_y) = y = \overline{\pi_1}(U_y)$; ie, W-almost everywhere $\overline{\pi_1} = \tau_U^{-1}$. If $\overline{\pi_1}(W) = V$, then $\overline{\pi_1}[W] = V$ [as $\{U_y : y \in B\} \in W$], so $\tau_U^{-1}[W] = V$, and $W = \tau_U[V]$. $\square$

Lemma 2.5 *$\langle u_i^{\bowtie} V, (u_i)_b, (u_{i+1})_b, \ldots, (u_\nu)_b\rangle$ is the u_i-pedigree for $\langle b, a\rangle$ over $B \times A$.*

Proof By Lemma 2.3, $\langle (u_i)_b, \ldots, (u_\nu)_b\rangle$ is a β-pedigree for $\langle b, a\rangle$ over $B \times A$. By the choice of b, $b\mathbb{M}_\gamma V$ holds for all $u_i \sqsubseteq \gamma \sqsubset \beta$. Hence $(u_i)_b = \tau_{u_i}(b)\mathbb{M}_\gamma \tau_{u_i}[V] = u_i^{\bowtie} V$ holds for all such γ [see the proof of Proposition 2.4]. As $u_i^{\bowtie} V \boxminus u_i$ and $(u_i)_b \boxminus b \boxminus \beta$, $\langle u_i^{\bowtie} V, (u_i)_b, \ldots, (u_\nu)_b\rangle$ is a u_i-pedigree. □

We can now complete the proof of Proposition 2.2.

Proof *of Proposition* 2.2 By Lemma 2.5 and RST, Corollary 10.6, the pedigree $\vec{v}$ for $\langle b, a\rangle$ over $B \times A$ has the form $\langle v_0, \ldots, v_m, (u_i)_b, \ldots, (u_\nu)_b\rangle$ where $v_m = u_i^{\bowtie} V$. For $b' \boxminus \beta'$ such that $u_i \sqsubset \beta' \sqsubset u_{i+1}$ [$u_\nu \sqsubset \beta'$ if $i = \nu$] and $b'\mathbf{M}_0 V$, the pedigree for $\langle b', a\rangle$ over $B \times A$ has the form $\vec{v}' = \langle v'_0, \ldots, v'_{m'}, (u_i)_{b'}, \ldots, (u_\nu)_{b'}\rangle$ where $v'_{m'} = u_i^{\bowtie} V$. However, it is trivial to verify that $\langle v_0, \ldots, v_m, (u_i)_{b'}, \ldots, (u_\nu)_{b'}\rangle$ is also a pedigree for $\langle b', a\rangle$ over $B \times A$. By the uniqueness of pedigrees [RST, Corollary 10.6], $m = m'$ and $v'_j = v_j$ for all $j \leq m$. □

Proof *of Theorem* 2.1 *(Strong Stability)* It suffices to give the proof for the case $\alpha \boxminus 0$; the general case follows by Transfer.

Let $L := \operatorname{ran} \vec{u}$ where $\vec{u} = \langle u_0, \ldots, u_\nu\rangle$ is the pedigree for $\langle \bar{y}\rangle \in A$ over some $A \in \mathbb{S}_0$, and let $\beta \cong_L \beta'$. If $\beta \boxminus \beta' \boxminus u_i$ for some $i \leq \nu$, then $\boldsymbol{\mathcal{P}}(\bar{x}, \bar{y}; \beta) \leftrightarrow \boldsymbol{\mathcal{P}}(\bar{x}, \bar{y}; u_i) \leftrightarrow \boldsymbol{\mathcal{P}}(\bar{x}, \bar{y}; \beta')$, as the variable z appears in $\boldsymbol{\mathcal{P}}(\bar{x}, \bar{y}; z)$ only in the scope of $\sqsubseteq$ [RST, Definition 12.25].

Assume now that $u_i \sqsubset \beta, \beta' \sqsubset u_{i+1}$ for some $i \leq \nu$ [$u_i \sqsubset \beta, \beta'$ if $i = \nu$]. Fix $V, b \in \mathbb{S}_0$, $V \in \boldsymbol{\beta}_1 B \setminus \boldsymbol{\beta}_0 B$, and b, b' such that $b \boxminus \beta$, $b\mathbf{M}_0 V$ and $b' \boxminus \beta'$, $b'\mathbf{M}_0 V$. By Proposition 2.2, the pedigrees for $\langle b, \langle \bar{y}\rangle\rangle$ and $\langle b', \langle \bar{y}\rangle\rangle$ have the form, respectively, $\langle v_0, \ldots, v_m, (u_i)_b, \ldots, (u_\nu)_b\rangle$ and $\langle v_0, \ldots, v_m, (u_i)_{b'}, \ldots, (u_\nu)_{b'}\rangle$.

To obtain the pedigrees for $\langle \langle \bar{x}\rangle, \langle b, \langle \bar{y}\rangle\rangle\rangle$ and $\langle \langle \bar{x}\rangle, \langle b', \langle \bar{y}\rangle\rangle\rangle$ where $\langle \bar{x}\rangle \in \mathbb{S}_0$, it is only necessary to subscript all terms of the above pedigrees by $\langle \bar{x}\rangle$ [Lemma 2.3]; in particular, both $\langle \langle \bar{x}\rangle, \langle b, \langle \bar{y}\rangle\rangle\rangle$ and $\langle \langle \bar{x}\rangle, \langle b', \langle \bar{y}\rangle\rangle\rangle$ have the same type $(v_0)_{\langle \bar{x}\rangle}$. It follows that $\langle \bar{x}, \bar{y}, b\rangle$ and $\langle \bar{x}, \bar{y}, b'\rangle$ have the same type. [Apply RST, Theorem 10.10 to the natural mapping $\pi : \langle \langle \bar{x}\rangle, \langle z, \langle \bar{y}\rangle\rangle\rangle \mapsto \langle \bar{x}, \bar{y}, z\rangle$ and its inverse.]

By RST, Theorem 12.11 (Normal Form Theorem), $\boldsymbol{\mathcal{P}}(\bar{x}, \bar{y}; b) \leftrightarrow \boldsymbol{\mathcal{P}}(\bar{x}, \bar{y}; b')$. As the variable z appears in the formula $\boldsymbol{\mathcal{P}}(\bar{x}, \bar{y}; z)$ only in the scope of $\sqsubseteq$, we have also $\boldsymbol{\mathcal{P}}(\bar{x}, \bar{y}; \beta) \leftrightarrow \boldsymbol{\mathcal{P}}(\bar{x}, \bar{y}; b)$ and $\boldsymbol{\mathcal{P}}(\bar{x}, \bar{y}; \beta') \leftrightarrow \boldsymbol{\mathcal{P}}(\bar{x}, \bar{y}; b')$. This proves the theorem. □

Stability for several levels simultaneously is expressed by Polytransfer [RST, Proposition 12.26; RST2, Proposition 1.10 (8)]. The argument in the previous proof can be pushed to establish a version of Strong Stability that generalizes Polytransfer.

Theorem 2.6 (**Strong Stability for several levels**) *Given $y_1, \ldots, y_\ell$, there is a level set L such that if $\alpha \sqsubset \beta_1 \sqsubset \ldots \sqsubset \beta_n$, $\alpha \sqsubset \beta'_1 \sqsubset \ldots \sqsubset \beta'_n$, and $\beta_i \cong_L \beta'_j$ for all $i, j \leq n$, then*

$$(\forall \bar{x} \in \mathbb{S}_\alpha)(\boldsymbol{\mathcal{P}}^\alpha(\bar{x}, \bar{y}; \beta_1, \ldots, \beta_n) \leftrightarrow \boldsymbol{\mathcal{P}}^\alpha(\bar{x}, \bar{y}; \beta'_1, \ldots, \beta'_n)).$$

Proof We give the proof for $\alpha \boxminus 0$ and $n = 2$. Using the notation from the proof of Theorem 2.1, let $u_i \sqsubset \beta_1, \beta_2, \beta'_1, \beta'_2 \sqsubset u_{i+1}$. Let $b_i \boxminus \beta_i$, $b_i \mathsf{M}_0 V$, $b'_i \boxminus \beta'_i$, $b'_i \mathsf{M}_0 V$, for $i = 1, 2$. By Proposition 2.2 applied twice, the pedigree for $\langle b_1, \langle b_2, \langle \bar{y} \rangle \rangle \rangle$ then has the form

$$\langle v_0, \ldots, v_m, (u_i^{\bowtie} V)_{b_1}, ((u_i)_{b_2})_{b_1}, \ldots, ((u_\nu)_{b_2})_{b_1} \rangle$$

and $v_0, \ldots, v_m = (u_i^{\bowtie} V)^{\bowtie} V$ are independent of the choice of $\beta_1, \beta_2, b_1, b_2$. It follows that $\langle \langle \bar{x} \rangle, \langle b_1, \langle b_2, \langle \bar{y} \rangle \rangle \rangle \rangle$ and $\langle \langle \bar{x} \rangle, \langle b'_1, \langle b'_2, \langle \bar{y} \rangle \rangle \rangle \rangle$ have the same type, namely $(v_0)_{\langle \bar{x} \rangle}$. The theorem follows from this observation. $\square$

Theorem 2.6 suggests a further generalization, in which the condition "$\beta_i \cong_L \beta'_j$ for all $i, j \leq n$" would be weakened to "$\beta_i \cong_L \beta'_i$ for all $i \leq n$." However, such generalization is false. Below, we give a counterexample.

First we need some observations. The technical notion of **good** α- pedigree is defined in RST, Definition 10.8. The following fact is used implicitly in RST.

Proposition 2.7 (**SST**$^\flat$; hence **GRIST**) *Every α-pedigree is good.*

Proof The axiom ($\mathbf{B}_\alpha$) [RST, page 65] implies that for every $y \in B \in \mathbb{S}_\alpha$ there is a good α-pedigree $\vec{v}$ for y over B. Indeed, fix $x \in A \in \mathbb{S}_\alpha$ where $x \in \mathbb{S}_\alpha$; then $\langle x \rangle$ is a good α-pedigree for $U = x$ over A, $\operatorname{Dom} U = A$, and $x \mathsf{M}_\alpha U$ holds. Let $F : B \to A$ be the constant function with value x; so $F \in \mathbb{S}_\alpha$ and $\operatorname{ran} F \subseteq \operatorname{Dom} U$. By ($\mathbf{B}_\alpha$) there is $V \in \mathbb{S}_\alpha$ with $U = \overline{F}(V)$ and $y \mathsf{M}_\alpha V$. The last statement asserts that there is a good α-pedigree $\vec{v} = \langle v_0, \ldots, v_\mu \rangle$ for y over some $B' \in \mathbb{S}_\alpha$ with $v_0 = V$. But, $\overline{F}(V)$ being defined implies $\operatorname{Dom} V = B$; as $\operatorname{Dom} v_0 = B'$, we have $B = B'$. If now $\vec{u}$ is any α-pedigree for y over B, then $\vec{u} = \vec{v}$ by RST, Proposition 10.3, so $\vec{u}$ is good. $\square$

Proposition 2.8 *If $\vec{u}$ is an α-pedigree (for x over $A \in \mathbb{S}_\alpha$), then*

$$(\mathbb{S}_\alpha[\vec{u}], =, \in, \sqsubseteq_\alpha) \vDash (\forall w)(\exists k \in \omega)(w \boxminus u_k).$$

Proof The claim follows from RST, Proposition 10.20. By RST, Definition 10.8 (and the fact that, in **GRIST**, all pedigrees are good), the interpretation $[\mathcal{U}\ell t(\mathbb{V}; \Sigma\, \mathbf{u})]^{\mathbb{S}_\alpha}$ is isomorphic to $\mathbb{S}_\alpha[\vec{u}]$ via $\mathrm{j} = \mathrm{j}_{\alpha;x,A}$, and $\mathrm{j}(D) = D(\vec{u}^+) = \vec{u}^+$, $\mathrm{j}(E) = E(\vec{u}^+) = x$, in the notation of RST, Proposition 10.20. By the same proposition (valid in **ZFC**),

$$\mathcal{U}\ell t(\mathbb{V}; \Sigma\, \mathbf{u}) \models (\forall f \sqsupset 0)(\exists k \in \omega)(f \boxminus D_k).$$

As $\mathbb{S}_\alpha \preccurlyeq \mathbb{I}$, the same is true in $[\mathcal{U}\ell t(\mathbb{V}; \Sigma\, \mathbf{u})]^{\mathbb{S}_\alpha}$, and via the isomorphism j, in $\mathbb{S}_\alpha[\vec{u}]$. Hence

$$\mathbb{S}_\alpha[\vec{u}] \models (\forall w \sqsupset 0)(\exists k \in \omega)(w \boxminus (\vec{u}^+)_k = u_{k+1}).$$

Of course, $\mathbb{S}_\alpha[\vec{u}] \models w \boxminus u_0$, if $w \boxminus 0$. □

Note: In RST, the notations $\mathcal{U}\ell t(\mathbb{V};\mathcal{U})$ and $\mathcal{U}\ell t(\mathbb{V};\Sigma\,\mathcal{U})$ are used interchangeably.

Proposition 2.9 *If $\vec{u} = \langle u_0, \dots, u_\nu\rangle$ is an α-pedigree and $\nu \in \mathbb{S}_\alpha$, then* $\operatorname{rank} u_0 = \nu$.

Proof It suffices to give a proof for $\alpha \boxminus 0$ [then use Transfer]. We prove that $\operatorname{rank} u_{\nu-i} = i$ by External Induction [RST, Proposition 12.19]. If $i = 0$, then $u_{\nu-i} = u_\nu = a \in \boldsymbol{\beta}_0 A$. If $u_{\nu-i} \in \boldsymbol{\beta}_i A \smallsetminus \boldsymbol{\beta}_{i-1} A$, then $u_{\nu-(i+1)}$ is nonprincipal and generated by $u_{\nu-i}$ over $\boldsymbol{\beta}_i A$, so it belongs to $\boldsymbol{\beta}_{i+1} A \smallsetminus \boldsymbol{\beta}_i A$. □

Construction of the example.

Let $A \in \mathbb{S}_0$ be an infinite set and $U \in \boldsymbol{\beta}_\omega A \smallsetminus \boldsymbol{\beta}_{<\omega} A$ be a stratified ultrafilter over A of rank ω. The axiom (**F**) [RST, page 66] guarantees that there is an $a \in A$ such that $a\mathsf{M}_0 U$ holds. We fix such an a and $V, B \in \mathbb{S}_0$, $V \in \boldsymbol{\beta}_1 B \smallsetminus \boldsymbol{\beta}_0 B$.

Consider the statement $\boldsymbol{\mathcal{P}}(A, V, B, a; \beta_1, \beta_2)$:

"*For every $b \in B$ such that $b \boxminus \beta_2$ and $b\mathsf{M}_0 V$, and for every pedigree $\vec{v} = \langle v_j : j \leq \mu\rangle$ for $\langle b, a\rangle$ over $B \times A$, there exists some $j \leq \mu$ such that $v_j \boxminus \beta_1$.*"

Let $L = \{w_0, w_1, \dots, w_\rho\}$ be any level set; wlog $w_0 \boxminus 0$. We find $\beta_1 \cong_L \beta_1'$ and β_2 such that $\boldsymbol{\mathcal{P}}(A, V, B, a; \beta_1, \beta_2)$ is true and $\boldsymbol{\mathcal{P}}(A, V, B, a; \beta_1', \beta_2)$ is false.

Let $\vec{u} = \langle u_0, u_1, \dots, u_\nu\rangle$ be the pedigree for a over A; we have $u_0 = U$. We note that $\nu \notin \mathbb{S}_0$; otherwise U would have rank $\nu \in \omega$, by Proposition 2.9.

Fix a level γ such that $0 \sqsubset \gamma \sqsubset u_1, w_1$. Fix $W \in \boldsymbol{\beta}_1 \omega \smallsetminus \boldsymbol{\beta}_0 \omega$ and $c \boxminus \gamma$, $c\mathsf{M}_0 W$. Then $c \in \omega$ and $c \notin \mathbb{S}_0$ because W is nonprincipal. We claim that $c < \nu$. Assume to the contrary that $\nu \leq c$, so $\nu \in \mathbb{S}_\gamma$. As $\langle u_0, \dots, u_\nu\rangle$ is also a γ-pedigree for a over A, by Proposition 2.9 $\operatorname{rank} U = \operatorname{rank} u_0 = \nu$ and $U \in \boldsymbol{\beta}_{<\omega} A$, a contradiction.

Fix β_2 such that $u_c \sqsubset \beta_2 \sqsubset u_{c+1}$. We observe that $b, a, \vec{u} \in \mathbb{S}_0[\vec{v}]$ [the last by RST, Corollary 10.18] and c is $\in$-definable from $\vec{u}$, $\vec{v}$ and b: c is the least j such that

$(u_j)_b = v_k$ for some $k \le \mu$. Hence $c \in \mathbb{S}_0[\vec{v}]$ and, by Proposition 2.8, $c \boxminus v_\ell$ for some $\ell > 0$.

The statement $\boldsymbol{\mathcal{P}}(A, V, B, a; \beta_1, \beta_2)$ holds for $\beta_1 := \gamma$ and β_2, because then $\beta_1 \boxminus c \boxminus v_\ell$.

On the other hand, by density of levels, there exist β_1' such that $0 \sqsubset \beta_1' \sqsubset v_1 \sqsubseteq v_\ell \boxminus \gamma \sqsubset w_1$. Then $\beta_1' \cong_L \beta_1$ and $\boldsymbol{\mathcal{P}}(A, V, B, a; \beta_1', \beta_2)$ fails. $\square$

This example can also be used to show that Polytransfer [RST, Proposition 12.26] does not hold for arbitrary finite (in the sense of **GRIST**) sequences of levels.

We say that sets S_1 and S_2 are **level–equivalent** if $(\forall x \in S_1)(\exists y \in S_2)(x \boxminus y)$ and $(\forall y \in S_2)(\exists x \in S_1)(x \boxminus y)$.

Claim *For A, U, a as above, the length ν of the pedigree $\vec{u}$ for a over A has itself a pedigree of the form $\langle v_0, v_1 \rangle$, where $v_1 \boxminus u_1$.*

Proof Refer to RST, Proposition 10.20. The interpretation $(\mathbb{S}_0[\vec{u}], =, \in, \sqsubseteq)$ is isomorphic to $[\mathcal{U}\ell t(\mathbb{V}; \Sigma\,\mathbf{U})]^{\mathbb{S}_0}$, and $\vec{v}^+$ corresponds to D in this isomorphism. This means that ν corresponds to the function $F : t \mapsto |t|$ defined for $t \in \Sigma\,\mathbf{U}$. Each such t has the form $\langle u \rangle \frown s$, for some $u \in U$. As the rank of U is ω, rank $u \in \omega$, and hence $|s| =$ rank u is independent of s. Define $f : [T_{\mathbf{U}}]_0 = U \to \omega$ by $f(u) = \operatorname{rank} u$; the above remarks show that $F(t) = f(u)$ holds for all t, ie, $\mathcal{U}\ell t(\mathbb{V}; \Sigma\,\mathbf{U}) \vDash F \boxminus D^1$. Hence $\mathcal{U}\ell t(\mathbb{V}; \Sigma\,\mathbf{U})$ satisfies the statement "The range of the pedigree for F is level–equivalent to $\{D^0, D^1\}$." As $\mathbb{S}_\alpha \preccurlyeq \mathbb{I}$, the same is true in $[\mathcal{U}\ell t(\mathbb{V}; \Sigma\,\mathbf{U})]^{\mathbb{S}_0}$. The isomorphism then establishes that $\mathbb{S}_0[\vec{u}]$ satisfies the statement "The range of the pedigree for ν is level–equivalent to $\{u_0, u_1\}$," and by RST, Proposition 10.17, the range of the pedigree for ν is level–equivalent to $\{u_0, u_1\}$. $\square$

Let now $\Lambda_1 = \{u_0 \sqsubset u_1 \sqsubset \ldots \sqsubset u_{\nu-1}\}$ and $\Lambda_2 = \{u_0 \sqsubset u_2 \sqsubset \ldots \sqsubset u_\nu\}$. The statement: "There is a pedigree $\vec{v}$ such that ran $\vec{v}$ is level–equivalent to Λ" is true about Λ_1 [$\langle u_0, u_1, \ldots, u_{\nu-1} \rangle$ is a pedigree for $u_{\nu-1} \in \boldsymbol{\beta}_1 A \in \mathbb{S}_0$], but false about Λ_2 [there can be no such pedigree, because ν would be $\in$-definable from it, and the range of the pedigree for ν is level–equivalent to $\{u_0, u_1\}$; but $u_1 \boxminus u_j$ does not hold for any $j \neq 1$, contradicting Proposition 2.8 and RST, Proposition 10.17].

A weaker generalization of Theorem 2.6 does hold. We state it only for $n = 2$; the higher values of n can be treated similarly.

Theorem 2.10 *Given $y_1, \ldots, y_\ell$, there is a level set L such that for every β_2 there is a level set M, dependent only on the $\cong_L$-equivalence class of β_2, such that if $\alpha \sqsubset \beta_1 \sqsubset \beta_2$, $\alpha \sqsubset \beta_1' \sqsubset \beta_2'$, $\beta_2 \cong_L \beta_2'$ and $\beta_1 \cong_M \beta_1'$, then*

$$(\forall \bar{x} \in \mathbb{S}_\alpha)(\boldsymbol{\mathcal{P}}^\alpha(\bar{x}, \bar{y}; \beta_1, \beta_2) \leftrightarrow \boldsymbol{\mathcal{P}}^\alpha(\bar{x}, \bar{y}; \beta_1', \beta_2')).$$

Proof Choose V, b_1, b_2, b'_1, b'_2 as in the proof of Theorem 2.6 (let $n = 2$). The pedigree for $\langle b_2, \langle\bar{y}\rangle\rangle$ is $\langle v_0, \dots, v_m, (u_i)_{b_2}, \dots, (u_\nu)_{b_2}\rangle$, where $v_m = u_i^{\bowtie}V$, so $v_m \boxminus u_i$, and $v_0, \dots, v_m$ depend only on the $\cong_L$-equivalence class of β_2. Let $M := \{v_0, \dots, v_m\}$.

If $v_j \sqsubset \beta_1 \sqsubset v_{j+1}$ [$v_j \sqsubset \beta_1$ if $j = m$], the pedigree for $\langle b_1, \langle b_2, \langle\bar{y}\rangle\rangle\rangle$ is

$$\langle w_0, \dots, w_p, (v_j)_{b_1}, (v_{j+1})_{b_1}, \dots, (v_m)_{b_1}, ((u_i)_{b_2})_{b_1}, \dots, ((u_\nu)_{b_2})_{b_1}\rangle,$$

where $w_p = v_j^{\bowtie}V$, so $w_p \boxminus v_j$, and $w_0, \dots, w_p$ are independent of the choice of b_1, b_2. Hence $\langle\langle\bar{x}\rangle, \langle b_1, \langle b_2, \langle\bar{y}\rangle\rangle\rangle\rangle$ and $\langle\langle\bar{x}\rangle, \langle b'_1, \langle b'_2, \langle\bar{y}\rangle\rangle\rangle\rangle$ have the same type $(w_0)_{\langle\bar{x}\rangle}$, and the theorem follows. $\square$

3 Miscellaneous other principles.

In this section we derive, in **GRIST**, several versions of Idealization and Choice that have been found useful in applications. In this section, $\mathcal{P}$ is always a $\mathbf{V}$-formula, unless stated otherwise.

Local Idealization combines Idealization and Local Transfer.

Local FRIST Idealization:
Given $\mathbf{U}$ and $x_1, \dots, x_n$, there is $\mathbf{V} \supset \mathbf{U}$ such that for all $A, B \in \mathbf{U}$,

$$(\forall a \in \mathcal{P}^{\mathbf{fin}} A \cap \mathbf{U})(\exists y \in B)(\forall x \in a)\mathcal{P}(x, y, \bar{x}; \mathbf{U}) \leftrightarrow (\exists y \in B)(\forall x \in A \cap \mathbf{U})\mathcal{P}(x, y, \bar{x}; \mathbf{V}).$$

Proof By Local Transfer there is $\mathbf{V} \supset \mathbf{U}$ such that for all $A, B \in \mathbf{U}$ and all $a \in \mathbf{U}$,

$$(\exists y \in B)(\forall x \in a)\mathcal{P}(x, y, \bar{x}; \mathbf{U}) \leftrightarrow (\exists y \in B)(\forall x \in a)\mathcal{P}(x, y, \bar{x}; \mathbf{V}).$$

Hence the left side of the claim is equivalent to

$$(\forall a \in \mathcal{P}^{\mathbf{fin}} A \cap \mathbf{U})(\exists y \in B)(\forall x \in a)\mathcal{P}(x, y, \bar{x}; \mathbf{V}),$$

which is equivalent to the right side by **FRIST** Idealization. $\square$

Local GRIST Idealization:
Given $\mathbf{U}$ and $x_1, \dots, x_n$, there is $\mathbf{V} \supset \mathbf{U}$ such that, for all $A \in \mathbf{U}$,

$$(\forall a \in \mathcal{P}^{\mathbf{fin}} A \cap \mathbf{U})(\exists y)(\forall x \in a)\mathcal{P}(x, y, \bar{x}; \mathbf{U}) \leftrightarrow (\exists y)(\forall \mathbf{W} \subset \mathbf{V})(\forall x \in A \cap \mathbf{W})\mathcal{P}(x, y, \bar{x}; \mathbf{V}).$$

Proof By Local Transfer there is $\mathbf{V} \supset \mathbf{U}$ such that, for all $A \in \mathbf{U}$,

$(\forall a \in \mathcal{P}^{\mathbf{fin}} A \cap \mathbf{U})(\exists y)(\forall x \in a)\boldsymbol{\mathcal{P}}(x, y, \bar{x}; \mathbf{U}) \leftrightarrow (\forall a \in \mathcal{P}^{\mathbf{fin}} A \cap \mathbf{V})(\exists y)(\forall x \in a)\boldsymbol{\mathcal{P}}(x, y, \bar{x}; \mathbf{V})$

and, for all finite $a \in \mathbf{U}$,

$$(\exists y)(\forall x \in a)\boldsymbol{\mathcal{P}}(x, y, \bar{x}; \mathbf{U}) \leftrightarrow (\exists y)(\forall x \in a)\boldsymbol{\mathcal{P}}(x, y, \bar{x}; \mathbf{V}).$$

Assume that $(\forall a \in \mathcal{P}^{\mathbf{fin}} A \cap \mathbf{U})(\exists y)(\forall x \in a)\boldsymbol{\mathcal{P}}(x, y, \bar{x}; \mathbf{U})$. Then in particular

$$(\forall \mathbf{W} \subset \mathbf{V})(\forall a \in \mathcal{P}^{\mathbf{fin}} A \cap \mathbf{W})(\exists y)(\forall x \in a)\boldsymbol{\mathcal{P}}(x, y, \bar{x}; \mathbf{V}),$$

which is the left side of **GRIST** Idealization. The right side follows.

Assume now that $(\exists y)(\forall \mathbf{W} \subset \mathbf{V})(\forall x \in A \cap \mathbf{W})\boldsymbol{\mathcal{P}}(x, y, \bar{x}; \mathbf{V})$, and let $\mathbf{W} = \mathbf{U}$ to obtain $(\exists y)(\forall x \in A \cap \mathbf{U})\boldsymbol{\mathcal{P}}(x, y, \bar{x}; \mathbf{V})$. It follows that $(\exists y)(\forall x \in a)\boldsymbol{\mathcal{P}}(x, y, \bar{x}; \mathbf{V})$ holds for all $a \in \mathcal{P}^{\mathbf{fin}} A \cap \mathbf{U}$ [because $a \subseteq A$, by RST2, Proposition 1.10 (13)], and hence $(\exists y)(\forall x \in a)\boldsymbol{\mathcal{P}}(x, y, \bar{x}; \mathbf{U})$ holds. □

We recall the principle of **Standard Size Choice** [RST2, Proposition 1.10 (10)]:
For every $A \in \mathbf{V}$ such that $(\forall x \in A \cap \mathbf{V})(\exists y)\boldsymbol{\mathcal{P}}(x, y, \bar{x}; \mathbf{V})$, there exists a function f with dom$f = A$ *such that* $(\forall x \in A \cap \mathbf{V})\boldsymbol{\mathcal{P}}(x, f(x), \bar{x}; \mathbf{V})$.

We call f a **choice function** for $\boldsymbol{\mathcal{P}}$ on A.

Strong Standard Size Choice: *For every $\mathbf{V} = \mathbf{V}(\alpha)$ there is $\mathbf{V}' \supset \mathbf{V}$ such that for all $A \in \mathbf{V}$ there exists a function f with* dom$f = A$ *such that, for all $x \in A$ with* $\mathbf{V}(x) \subset \mathbf{V}'$, $(\exists y)\boldsymbol{\mathcal{P}}(x, y, \bar{x}; \mathbf{V}(\alpha, x)) \to \boldsymbol{\mathcal{P}}(x, f(x), \bar{x}; \mathbf{V}(\alpha, x))$.

The level $\mathbf{V}'$ depends on the parameters $\bar{x}$. If $\mathbf{V}'$ were allowed to depend also on f, the principle would follow easily from Standard Size Choice and Local Transfer. To prove it as is, we follow the argument of RST, Proposition 12.28 (α-Standard Size Choice) with minor changes, except for the proof of Claim, where the use of Local Transfer has to be replaced by an appeal to Strong Stability.

Proof We first reformulate Strong Standard Size Choice in the $\in$-$\sqsubseteq$–language.

Let $\boldsymbol{\mathcal{P}}(x, y, \bar{x})$ be an $\in$-$\sqsubseteq$-formula. For every α there is $\beta \sqsupset \alpha$ such that for all $A \sqsubseteq \alpha$ there is a function f with dom$f \subseteq A$ *such that, for all $x \in A$, $x \sqsubset \beta$,* $(\exists y)\boldsymbol{\mathcal{P}}^{\langle \alpha, x\rangle}(x, y, \bar{x}) \to \boldsymbol{\mathcal{P}}^{\langle \alpha, x\rangle}(x, f(x), \bar{x})$.

We fix $\bar{x}$ and α. Let $\boldsymbol{\mathcal{P}}'(z, y)$ be the formula $(\exists x, \bar{x})[z = \langle x, \langle \bar{x}\rangle\rangle \ \wedge \ \boldsymbol{\mathcal{P}}(x, y, \bar{x})]$, and let $\boldsymbol{\mathcal{Q}}(V)$ be the $\in$-formula corresponding to $\boldsymbol{\mathcal{P}}'$ by the Normal Form Theorem. If $\alpha \sqsubseteq \gamma$, $\langle \bar{x}\rangle \mathbf{M}_\gamma U$ and $x \in \mathbb{S}_\gamma$, we then have

$$(\exists y)\boldsymbol{\mathcal{P}}^{\gamma}(x, y, \bar{x}) \leftrightarrow (\exists V)[\overline{\pi_1}(V) = U_x \ \wedge \ \boldsymbol{\mathcal{Q}}(V)].$$

The **ZFC** principle of Selection: *Let* $\mathcal{R}(x, y, \bar{p})$ *be an* $\in$*-formula;* $(\forall \bar{p})(\forall A)[(\forall x \in A)(\exists y)\mathcal{R}(x, y, \bar{p}) \to (\exists f)(f \text{ is a function } \wedge (\forall x \in A)\mathcal{R}(x, f(x), \bar{p}))]$, which holds in $(\mathbb{S}_\alpha, \in)$, implies that for every $A \sqsubseteq \alpha$ there are functions $V, B \in \mathbb{S}_\alpha$ such that $\operatorname{dom} V = \operatorname{dom} B = \{x \in A : (\exists V)[\overline{\pi_1}(V) = U_x \wedge \mathcal{Q}(V)]\}$, and for all $x \in \operatorname{dom} V$, $V(x) \in \boldsymbol{\beta}_\infty B(x) \wedge \overline{\pi_1}(V(x)) = U_x \wedge \mathcal{Q}(V(x))$. It remains to prove the following.

Claim. *There is a level* $\beta \sqsupset \alpha$*, and for every* $A \sqsubseteq \alpha$ *there is a function* $\vec{v}$ *with* $\operatorname{dom} \vec{v} = \operatorname{dom} V$ *such that for all* $x \in \operatorname{dom} \vec{v}$*,* $x \sqsubset \beta$*,* $\vec{v}(x) = \langle v(x)_0, \ldots, v(x)_{\nu(x)} \rangle$ *is an* $\langle \alpha, x \rangle$*-pedigree over* $B(x)$ *with* $v(x)_0 = V(x)$ *and* $v(x)_{\nu(x)} = \langle \langle x, \langle \bar{x} \rangle \rangle, y(x) \rangle$ *for some (uniquely determined)* $y(x)$*.*

The function f on A defined on $\operatorname{dom} V$ by $x \mapsto y(x)$ then has the property that, for all $x \sqsubset \beta$ such that $(\exists y)\mathcal{P}^{\langle \alpha, x \rangle}(x, y, \bar{x})$, $\langle \langle x, \langle \bar{x} \rangle \rangle, f(x) \rangle \mathbf{M}_{\langle \alpha, x \rangle} V(x) \wedge \mathcal{Q}(V(x))$; so $(\mathcal{P}')^{\langle \alpha, x \rangle}(\langle x, \langle \bar{x} \rangle \rangle, f(x))$ holds, ie, $\mathcal{P}^{\langle \alpha, x \rangle}(x, f(x), \bar{x})$ holds.

Proof of Claim.

Let $\vec{u} = \langle u_0, u_1, \ldots, u_\mu \rangle$ be an α-pedigree for $\langle \bar{x} \rangle$. We fix a level β such that $\alpha \sqsubset \beta \sqsubset u_1$ [$\alpha \sqsubset \beta$ if $\mu = 0$]. Let $x \in \operatorname{dom} V$, $x \sqsubset \beta$.

By Lemma 2.3, $\vec{u}_x := \langle (u_0)_x, (u_1)_x, \ldots, (u_\mu)_x \rangle$ is an $\langle \alpha, x \rangle$-pedigree for $\langle x, \langle \bar{x} \rangle \rangle$, and $(u_j)_x \boxminus_{\langle \alpha, x \rangle} u_j$ for all $j \le \mu$. Let $\vec{v} = \langle v_0, v_1, \ldots, v_{\nu(x)} \rangle$ be some $\langle \alpha, x \rangle$-pedigree over $B(x)$ with $v_0 = V(x)$ and $v_{\nu(x)} = \langle \langle x, \langle \bar{x} \rangle \rangle, y \rangle$, for some y. If $\beta \sqsubset v_1$ [or if $\mu = 0$], then $\vec{v}$ is also a β-pedigree. If $v_1 \boxminus \gamma \sqsubseteq \beta$, fix δ so that $\beta \sqsubset \delta \sqsubset u_1$ [$\beta \sqsubset \delta$ if $\mu = 0$] (density). Then $\gamma \cong_L \delta$ for $L := \operatorname{ran} \vec{u}_x$, so by Strong Stability, there is an $\langle \alpha, x \rangle$-pedigree $\vec{v}$ as above with $v_1 \boxminus \delta \sqsupset \beta$. We again conclude that $\vec{v}$ is a β-pedigree.

The above argument shows that for every $x \in \operatorname{dom} V$ there is a β-pedigree $\vec{v}$ over $B(x)$ with $v_0 = V(x)$ and $v_{\nu(x)} = \langle \langle x, \langle \bar{x} \rangle \rangle, y \rangle$ for some y.

Similar to the proof of RST, Proposition 12.28, using α-Finite Choice and **GRIST** Idealization we obtain a function $\vec{v}$ with $\operatorname{dom} \vec{v} = \operatorname{dom} V$ such that for every $x \in \operatorname{dom} V$, $x \sqsubset \beta$, $\vec{v}(x)$ is a β-pedigree with the required properties. As $v(x)_0 = V(x) \sqsubseteq \langle \alpha, x \rangle$ and $v(x)_1 \sqsupset \beta \sqsupset \langle \alpha, x \rangle$ [if $\nu(x) > 0$], $\vec{v}(x)$ is an $\langle \alpha, x \rangle$-pedigree. □ □

The last principle we consider was proposed by Andreev [1] in the context of **IST**. It is a strengthening of the well-known Robinson's Lemma. The most important case is $A = \mathbb{N}$. If $\mathcal{Q}(n, y, \bar{z})$ is an internal statement [RST2, page 12; [6]], F is a function on $\mathbb{N}$, and $\mathcal{Q}(n, F(n), \bar{z})$ is valid for all $n \in \mathbf{V}$, then by overflow it remains valid for some, but not necessarily all, $n \notin \mathbf{V}$ as well. Andreev Principle says that we can do better, at least for internal statements with parameters from $\mathbf{V}$.

Definition 3.1 A function F with $\operatorname{dom} F = A \in \mathbf{V}$ is called **adequate** relative to $\mathbf{V}$ if $(\forall X \in \mathbf{V})(F \restriction \mathbf{V} \subseteq X \to F \subseteq X)$.

The usefulness of adequate functions comes from the following observation.

Let $\mathcal{Q}(x, y, \bar{z})$ be any internal formula. If $\bar{z} \in \mathbf{V}$ and $\mathcal{Q}(x, F(x), \bar{z})$ holds for all $x \in A \cap \mathbf{V}$, then it holds for all $x \in A$.

Proof Fix $B \in \mathbf{V}$ such that $\operatorname{ran} F \subseteq B$ and let $X = \{\langle x, y\rangle \in A \times B : \mathcal{Q}(x, y, \bar{z})\}$. Then $X \in \mathbf{V}$ and the definition of adequacy for this X translates into the preceding observation. □

Theorem 3.2 *For every function G defined on $A \in \mathbf{V}$ there exists an adequate function F defined on A such that $F \restriction \mathbf{V} = G \restriction \mathbf{V}$.*

Proof Let $G : A \to B$ be given, with $A, B \in \mathbf{V}$. By **FRIST** Standardization, there is a set $K \in \mathbf{V}$, $K \subseteq \mathcal{P}(A \times B)$, such that for all $X \subseteq A \times B$, $X \in \mathbf{V}$,

$$X \in K \leftrightarrow (\forall x \in A \cap \mathbf{V})(\langle x, G(x)\rangle \in X).$$

We note that $X_1, X_2 \in K \cap \mathbf{V}$ implies $X_1 \cap X_2 \in K$, hence, by Transfer, K is closed under finite intersections. Also, $(\forall x \in A)(\exists y)(\langle x, y\rangle \in X)$ holds for all $X \in K \cap \mathbf{V}$, hence, by Transfer, for all $X \in K$.

For every $a \in A$ let $L_a = \{\langle x, y\rangle \in A \times B : x = a \to y = G(a)\}$.

It is now clear that for every finite $\{X_1, \ldots, X_n\} \subseteq K \cap \mathbf{V}$ and $\{a_1, \ldots, a_m\} \subseteq A \cap \mathbf{V}$ there is a function $F : A \to B$ such that

$$F \subseteq X_1 \cap \ldots \cap X_n \cap L_{a_1} \cap \ldots \cap L_{a_m}.$$

By Saturation (Idealization) there is F such that $F \subseteq X \cap L_a$ for all $X \in K \cap \mathbf{V}$ and $a \in A \cap \mathbf{V}$. Clearly F is adequate and $F \restriction A = G \restriction A$. □

Corollary 3.3 (Andreev Principle) *Every external function defined on $A \cap \mathbf{V}$ has an extension to a function F defined on A such that if $\mathcal{Q}(x, y, \bar{z})$ is any internal formula, $\bar{z} \in \mathbf{V}$ and $\mathcal{Q}(x, F(x), \bar{z})$ holds for all $x \in A \cap \mathbf{V}$, then it holds for all $x \in A$.*

A typical application of Andreev Principle.

Let $\langle a_n : n \in \mathbb{N}\rangle$ be an adequate sequence where each $a_n \simeq 0$ relative to $\mathbf{V}$, for $n \in \mathbf{V}$. Then $\Sigma_{n=0}^{\infty} |a_n| \simeq 0$ relative to $\mathbf{V}$.

Proof For any fixed $\varepsilon > 0$, $\varepsilon \in \mathbf{V}$, the statement $\mathcal{Q}(n, a, \varepsilon) : |a| < \varepsilon/2^n$ is internal and $|a_n| < \varepsilon/2^n$ holds for all $n \in \mathbb{N} \cap \mathbf{V}$. Hence, $|a_n| < \varepsilon/2^n$ holds for all $n \in \mathbb{N}$. It follows that $\Sigma_{n=0}^{\infty} |a_n| < 2\varepsilon$. As $\varepsilon \in \mathbf{V}$ is arbitrary, $\Sigma_{n=0}^{\infty}|a_n| \simeq 0$. □

4 Relative continuity

Definition 4.1 (RST2, Definition 2.17) The function f is (uniformly) $(\mathbf{V}_1, \mathbf{V}_2)$–**continuous** if $x \simeq_{\mathbf{V}_1} x'$ implies $f(x) \simeq_{\mathbf{V}_2} f(x')$, for all $x, x' \in \operatorname{dom} f$. The function f is $\mathbf{V}$–**continuous** if it is $(\mathbf{V}, \mathbf{V})$-continuous.

RST2, Theorem 2.10 shows that for every function f there is a finite set $\{v_0, \ldots, v_n\}$ such that $\mathbf{V}(\cdot) = \mathbf{V}(v_0) \subset \mathbf{V}(v_1) \subset \ldots \subset \mathbf{V}(v_n)$ and, for all $\mathbf{V}(v_i) \subseteq \mathbf{V} \subset \mathbf{V}(v_{i+1})$ [all $\mathbf{V}(v_i) \subseteq \mathbf{V}$ if $i = n$], the function f is $\mathbf{V}(v_i)$–continuous if and only if f is $\mathbf{V}$–continuous if and only if f is not $\mathbf{V}(v_{i+1})$–continuous. Here we study $(\mathbf{V}_1, \mathbf{V}_2)$–continuity for $\mathbf{V}_1 \neq \mathbf{V}_2$. We consider only functions $f : [a, b] \to [c, d]$ where $a, b, c, d \in \mathbf{V}(\cdot)$, for simplicity.

(I) $(\mathbf{V}_1, \mathbf{V}_2)$–continuity for $\mathbf{V}_1 \subset \mathbf{V}_2$.

Definition 4.2 The function f is $\mathbf{V}$–**constant** if $(\forall x, y \in \operatorname{dom} f)(f(x) \simeq_{\mathbf{V}} f(y))$.

Proposition 4.3 *If $\mathbf{V}_1 \subset \mathbf{V}_2$ and f is $(\mathbf{V}_1, \mathbf{V}_2)$–continuous, then f is $\mathbf{V}_2$–constant.*

Proof Fix a natural number $N \in \mathbf{V}_2$, $N \notin \mathbf{V}_1$. For $x, y \in [a, b]$, $x < y$, let $\Delta x := (y - x)/N$ and $x_i := x + i \cdot \Delta x$ for $i \leq N$. Then $\Delta x \simeq_{\mathbf{V}_1} 0$, so $x_i \simeq_{\mathbf{V}_1} x_{i+1}$, and $f(x_i) \simeq_{\mathbf{V}_2} f(x_{i+1})$ by $(\mathbf{V}_1, \mathbf{V}_2)$–continuity. Let $\varepsilon := \max\{| f(x_{i+1}) - f(x_i) | : i < N\}$; note that $\varepsilon \simeq_{\mathbf{V}_2} 0$. We have $| f(y) - f(x) | \leq \Sigma_{i<N} | f(x_{i+1}) - f(x_i) | \leq \varepsilon \cdot N \simeq_{\mathbf{V}_2} 0$, because $\varepsilon \simeq_{\mathbf{V}_2} 0$ and $N \in \mathbf{V}_2$. Hence $f(x) \simeq_{\mathbf{V}_2} f(y)$. □

Corollary 4.4 *If f is $(\mathbf{V}_1, \mathbf{V}_2)$–continuous for $\mathbf{V}_1 \subset \mathbf{V}_2$, then f is $(\mathbf{V}, \mathbf{V}_2)$–continuous for all $\mathbf{V}$, and $\mathbf{V}$–continuous for all $\mathbf{V} \subseteq \mathbf{V}_2$.*

If f is $\mathbf{V}$–constant for all $\mathbf{V}$, then f is constant. Otherwise, there is a coarsest level $\mathbf{V}$ such that f is not $\mathbf{V}$–constant [by Granularity]; we denote it $\mathbf{V}_f$. Then f is $(\mathbf{V}_1, \mathbf{V}_2)$–continuous for all $\mathbf{V}_1$ and all $\mathbf{V}_2 \subset \mathbf{V}_f$, and is not $(\mathbf{V}_1, \mathbf{V}_2)$–continuous for any $\mathbf{V}_1 \subset \mathbf{V}_2$ when $\mathbf{V}_f \subseteq \mathbf{V}_2$. These observations completely describe the behavior of $(\mathbf{V}_1, \mathbf{V}_2)$–continuity for the case $\mathbf{V}_1 \subset \mathbf{V}_2$.

(II) $(\mathbf{V}_1, \mathbf{V}_2)$–continuity for $\mathbf{V}_2 \subset \mathbf{V}_1$.

If f is $\mathbf{V}$–continuous for some $\mathbf{V}$ such that $\mathbf{V}_2 \subseteq \mathbf{V} \subseteq \mathbf{V}_1$, then trivially it is $(\mathbf{V}_1, \mathbf{V}_2)$–continuous. Hence it remains to investigate the case when $\mathbf{V}(v_j) \subseteq \mathbf{V}_2 \subset \mathbf{V}_1 \subset \mathbf{V}(v_{j+1})$ and f is $\mathbf{V}$–discontinuous for all $\mathbf{V}(v_j) \subseteq \mathbf{V} \subset \mathbf{V}(v_{j+1})$.

Let us fix $\mathbf{V}_2$ such that $\mathbf{V}(v_j) \subseteq \mathbf{V}_2 \subset \mathbf{V}(v_{j+1})$. Since f is $\mathbf{V}_2$–discontinuous,

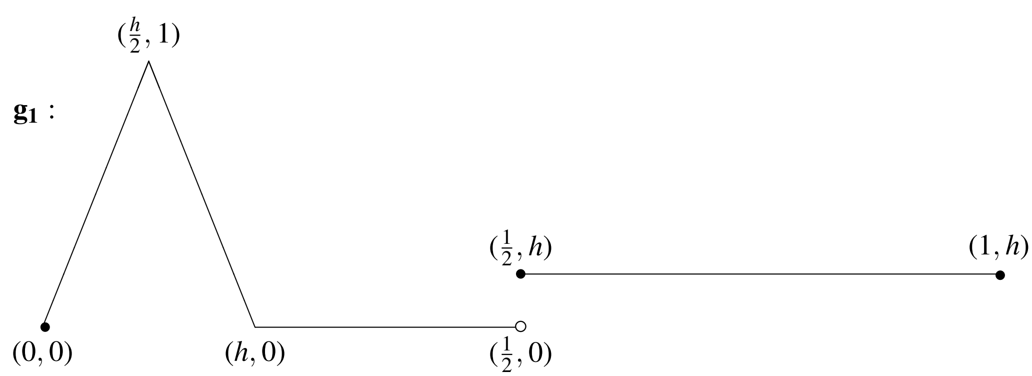
$$(\exists x, x' \in [a,b])(x \simeq_{\mathbf{V}_2} x' \ \wedge \ |f(x) - f(x')| > \varepsilon)$$

holds for some $\varepsilon > 0$, $\varepsilon \in \mathbf{V}_2$.

By Local Transfer, $(\exists x, x' \in [a,b])(x \simeq_{\mathbf{V}} x' \ \wedge \ |f(x)-f(x')| > \varepsilon)$ is then true for some $\mathbf{V} \supset \mathbf{V}_2$, so f is also $(\mathbf{V}, \mathbf{V}_2)$–discontinuous. This argument also shows that there is no finest level $\mathbf{V}$ such that f is $(\mathbf{V}, \mathbf{V}_2)$–discontinuous.

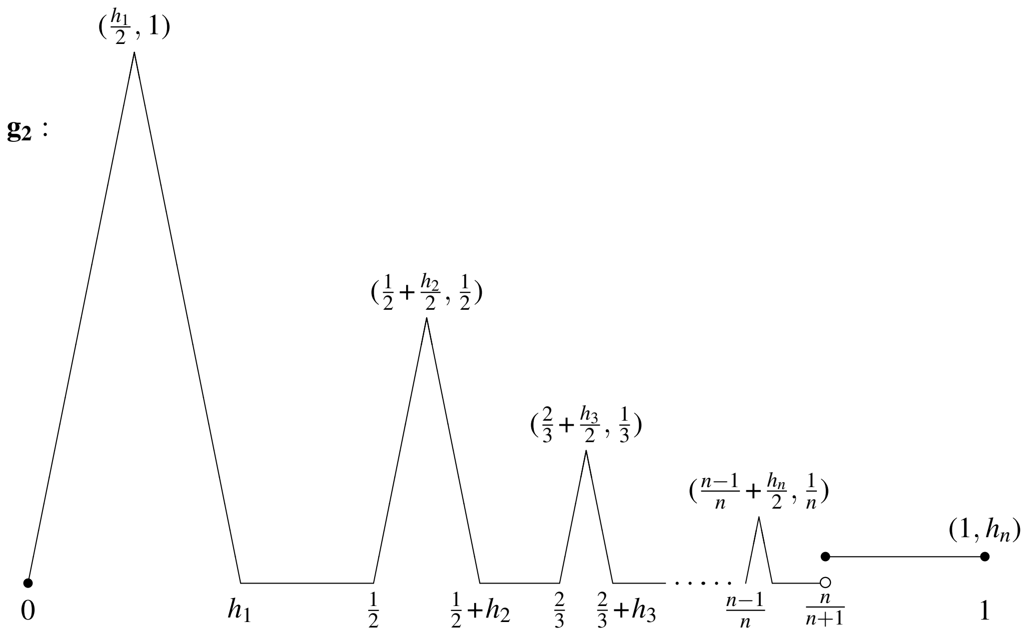
Each such $\mathbf{V}_2$ determines a "cut" in the ordering of levels by $\subseteq$. The "lower class" of the "cut" consists of those $\mathbf{V}$ for which f is $(\mathbf{V}, \mathbf{V}_2)$–discontinuous, and the "upper class" of those for which f is $(\mathbf{V}, \mathbf{V}_2)$–continuous (the latter can be empty). These "cuts" increase with $\mathbf{V}_2$: If $\mathbf{V}_2 \subseteq \mathbf{V}_2'$ and f is $(\mathbf{V}, \mathbf{V}_2)$–discontinuous, then f is $(\mathbf{V}, \mathbf{V}_2')$–discontinuous. The "upper class" may or may not have a coarsest "element," as the two following examples show.

Example 1. Let $h \simeq_{\mathbf{V}} 0$ for all $\mathbf{V} \subset \mathbf{V}(h) \supset \mathbf{V}(\cdot)$ [see RST2, Corollary 2.16].



The function g_1 is $\mathbf{V}$–discontinuous for all $\mathbf{V}$. If $\mathbf{V} \subset \mathbf{V}(h)$, then g_1 is $(\mathbf{V}, \mathbf{V}(\cdot))$–discontinuous. If $\mathbf{V}(h) \subseteq \mathbf{V}$, then g_1 is $(\mathbf{V}, \mathbf{V}(\cdot))$–continuous.

Example 2. Let $\{v_0, \ldots, v_n\}$ be a level set with $n \notin \mathbf{V}(\cdot)$, and $\{h_0, \ldots, h_n\}$ be as in RST2, Corollary 2.16; ie, $\mathbf{V}(h_i) = \mathbf{V}(v_i)$ and h_i is ultrasmall relative to all $\mathbf{V} \subset \mathbf{V}(h_i)$.



If $\mathsf{V} \subset \mathsf{V}(h_i)$ for some $i \in \mathsf{V}(\cdot)$, then g_2 is not $(\mathsf{V}, \mathsf{V}(\cdot))$–continuous, because of the spike at $1 - 1/i$, of width $h_i \simeq_{\mathsf{V}} 0$ and height $1/i \in \mathsf{V}(\cdot)$.

Otherwise, $\mathsf{V}(h_i) \subseteq \mathsf{V}$ for some $i \notin \mathsf{V}(\cdot)$ [see Proposition 6.2]. As $\{h_0, \ldots, h_{i-1}\} \in \mathsf{V}(h_{i-1})$, the function $g_2 \restriction [0, 1 - 1/i] \in \mathsf{V}(h_{i-1})$ [it is definable in $\mathsf{V}(h_{i-1})$] and is continuous, ie, $\mathsf{V}(h_{i-1})$–continuous, hence $(\mathsf{V}, \mathsf{V}(\cdot))$–continuous. The function $g_2 \restriction [1-1/i, 1]$ is also $(\mathsf{V}, \mathsf{V}(\cdot))$–continuous, because it is bounded by $\max\{\frac{1}{i}, h_n\} \simeq_{\mathsf{V}(\cdot)} 0$. Hence g_2 is $(\mathsf{V}, \mathsf{V}(\cdot))$–continuous.

Finally, we note that g_2 is not V–continuous for any V, because of the spike at $1 - \frac{1}{i+1}$ if $\mathsf{V}(h_i) \subseteq \mathsf{V} \subset \mathsf{V}(h_{i+1})$, and because of the jump at $1 - \frac{1}{n+1}$ if $\mathsf{V}(h_n) \subseteq \mathsf{V}$.

5 Conservativity of variants of relative set theory over BST.

In **GRIST**, the ordering of levels by inclusion is dense. In contrast, **Discrete GRIST** [see RST, Section 12] postulates that the ordering of levels by inclusion is discrete. O'Donovan raised a question of the extent to which these two theories agree. On the one hand, they both are conservative extensions of **ZFC**, and hence they prove the same $\in$-statements (namely, exactly those that are provable in **ZFC**). On the other hand, clearly there are statements about levels where the two theories differ. In this section we obtain a general result showing that provable statements in which no quantification over levels occurs are exactly the same in **GRIST**, **Discrete GRIST**, and other similar theories.

The theory **BST** is a modification of Nelson's **IST** [8], introduced by Kanovei (see Kanovei–Reeken [7] and RST, Section 5). The language of **BST** has $\in$ and a unary predicate $\mathbf{st}(\cdot)$ (" $\cdot$ *is standard* ").

If $\mathcal{P}(x_1, \ldots, x_n)$ is a formula of the $\in$-**st**-language, we let $\mathcal{P}(x_1, \ldots, x_n \mid \mathsf{V})$ be the formula obtained from $\mathcal{P}(\overline{x})$ by replacing each occurrence of $\mathbf{st}(x)$ by $x \in \mathsf{V}$. We note that $\mathcal{P}(\overline{x} \mid \mathsf{V})$ is a V-formula in the sense of RST2, as no quantification over levels occurs in it. Conversely, every formula $\mathcal{Q}(\overline{x}, \mathsf{V})$ of the language of **GRIST** in which no quantification over levels occurs is of the form $\mathcal{P}(\overline{x} \mid \mathsf{V})$ for some $\in$-**st**-formula $\mathcal{P}(\overline{x})$. The formula $\mathcal{P}^{\mathsf{W}}(\overline{x} \mid \mathsf{V})$ is obtained from $\mathcal{P}(\overline{x} \mid \mathsf{V})$ by replacing each occurence of $(\forall x)(\ldots)$ by $(\forall x \in \mathsf{W})(\ldots)$ and each $(\exists x)(\ldots)$ by $(\exists x \in \mathsf{W})(\ldots)$.

Let $\mathcal{T}$ be a theory in the language of **GRIST**. We say that $\mathcal{T}$ is **locally BST** if

(0) $\mathcal{T} \vdash \mathbf{R}^{\heartsuit}(\text{o} - \text{iv})$ [see RST2], ie, $\mathcal{T}$ proves that levels are linearly ordered by inclusion and there is a coarsest level $\mathbb{S} := \mathsf{V}(0) = \mathsf{V}(\cdot)$;

(1) $\mathcal{T} \vdash \mathcal{P}(\overline{x} \mid \mathsf{V})$ and $\mathcal{T} \vdash \mathsf{V}_1 \subset \mathsf{V}_2 \to \mathcal{P}^{\mathsf{V}_2}(\overline{x} \mid \mathsf{V}_1)$,
where $\mathcal{P}(\overline{x})$ is any axiom of **BST**; and

(2) For every countable model $\mathbf{M}$ of **ZFC** there is a countable model $\mathbf{N}$ of $\mathcal{T}$ such that $\mathbf{M}$ is isomorphic to $\mathbf{N} \restriction \mathbb{S}^{\mathbf{N}}$.

Condition (2) holds if $\mathcal{T}$ is realistic over **ZFC** in the sense of RST, Section 1. The theories **GRIST**, **Discrete GRIST**, and **FRIST** (more precisely, the theory called $\mathbf{FRBST}_2$ in [2]) are all locally **BST**. It is also easy to formulate a "bounded" version of **RIST** which is locally **BST**.

Theorem 5.1 *If $\mathcal{T}$ is locally* **BST**, *then $\mathcal{T} \vdash \mathcal{P}(\overline{x} \mid \mathsf{V})$ if and only if* **BST** $\vdash \mathcal{P}(\overline{x})$.

Proof The "if" direction is an immediate consequence of condition (1) [and the fact that the translation $\mathcal{P}(\overline{x}) \mapsto \mathcal{P}(\overline{x} \mid \mathsf{V})$ preserves logical axioms and deduction rules].

For the "only if" direction, assume that **BST** $\nvdash \mathcal{P}(\overline{x})$. Then there is a countable model $\mathbf{P}$ of **BST** $\wedge\, (\exists \overline{x}) \neg \mathcal{P}(\overline{x})$. Let $\mathbf{M} := \mathbf{P} \restriction \mathbb{S}^{\mathbf{P}}$, where $\mathbb{S} := \{x : \mathbf{st}(x)\}$. Condition (2) implies that there is a countable model $\mathbf{N}$ of $\mathcal{T}$ with $\mathbf{N} \restriction \mathbb{S}^{\mathbf{N}}$ isomorphic to $\mathbf{M}$. By condition (1), $\overline{\mathbf{P}} := (|\mathbf{N}|, \in, \mathbb{S}^{\mathbf{N}})$ is a model of **BST**. In RST, Corollary 5.9, it is proved that if $\mathbf{P}$, $\overline{\mathbf{P}}$ are countable models of **BST** with isomorphic standard universes, then $\mathbf{P}$ and $\overline{\mathbf{P}}$ are isomorphic. Hence $\overline{\mathbf{P}} \models (\exists \overline{x}) \neg \mathcal{P}(\overline{x})$ and $\mathbf{N} \models (\exists \overline{x}) \neg \mathcal{P}(\overline{x} \mid \mathbb{S})$. This shows $\mathcal{T} \nvdash \mathcal{P}(\overline{x} \mid \mathsf{V})$. $\square$

Theorem 5.1 can be extended to formulas with any finite list of levels. Let $\mathbf{BST}_k$ be the theory in the language with $\in$ and unary predicates $\mathbf{st}_1, \ldots, \mathbf{st}_k$, postulating

(i) $\mathbf{st}_1(x) \to \mathbf{st}_2(x) \to \ldots \to \mathbf{st}_k(x)$, and

(ii) $\mathcal{P}^i(\bar{x})$ for $1 \leq i \leq k$, where $\mathcal{P}(\bar{x})$ is any axiom of **BST**, and $\mathcal{P}^i(\bar{x})$ is obtained from $\mathcal{P}(\bar{x})$ by replacing each occurence of $\mathbf{st}(x)$ by $\mathbf{st}_i(x)$ and, if $i < k$, each $(\forall x)(\ldots)$ by $(\forall x)(\mathbf{st}_{i+1}(x) \to \ldots)$ and each $(\exists x)(\ldots)$ by $(\exists x)(\mathbf{st}_{i+1}(x) \wedge \ldots)$.

We note that $\mathbf{BST}_1$ is **BST** (with **st** replaced by $\mathbf{st}_1$) and $\mathbf{BST}_k$ is "**BST** iterated k times." It follows immediately by induction that if $\mathbf{P}$, $\overline{\mathbf{P}}$ are countable models of $\mathbf{BST}_k$ with isomorphic standard universes [ie, $\mathbf{P} \restriction \{x \in |\mathbf{P}| : \mathbf{P} \models \mathbf{st}_1(x)\}$ is isomorphic to $\overline{\mathbf{P}} \restriction \{x \in |\overline{\mathbf{P}}| : \overline{\mathbf{P}} \models \mathbf{st}_1(x)\}$, then $\mathbf{P}$ and $\overline{\mathbf{P}}$ are isomorphic.

Let $\mathcal{P}(\bar{x} \mid \mathbf{V}_1, \ldots, \mathbf{V}_k)$ be the formula obtained from $\mathcal{P}$ by replacing each occurence of $\mathbf{st}_i$ by $x \in \mathbf{V}_i$, for all $1 \leq i \leq k$. The argument in the proof of Theorem 5.1, with obvious modifications, proves the following theorem.

Theorem 5.2 *If $\mathcal{T}$ is locally* **BST**, *then*

$\mathcal{T} \vdash \mathbf{V}_1 \subset \mathbf{V}_2 \subset \ldots \subset \mathbf{V}_k \to \mathcal{P}(\bar{x} \mid \mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_k)$ *if and only if* $\mathbf{BST}_k \vdash \mathcal{P}(\bar{x})$.

Corollary 5.3 *If $\mathcal{T}$ is locally* **BST**, *then* $\mathcal{T} \vdash \mathcal{P}(\bar{x} \mid \mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_k)$ *if and only if* $\mathbf{GRIST} \vdash \mathcal{P}(\bar{x} \mid \mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_k)$.

Proof For $k = 2$ the formula $\mathcal{P}(\bar{x} \mid \mathbf{V}_1, \mathbf{V}_2)$ is equivalent to

$$(\mathbf{V}_1 \subset \mathbf{V}_2 \to \mathcal{P}(\bar{x} \mid \mathbf{V}_1, \mathbf{V}_2)) \wedge (\mathbf{V}_2 \subset \mathbf{V}_1 \to \mathcal{P}(\bar{x} \mid \mathbf{V}_1, \mathbf{V}_2)) \wedge \mathcal{P}(\bar{x} \mid \mathbf{V}_1, \mathbf{V}_1).$$

The claim follows from Theorems 5.2 and 5.1. Similarly for $k > 2$. □

6 Corrections and additions to RST.

The simplified proof of Proposition 6.10 in RST given there does not establish that $T' \preccurlyeq_{\mathcal{U}} T$. Below we give the original inductive proof.

Proposition 6.1 (RST, Proposition 6.10) *For every $\lambda : \subseteq \Sigma T \to \omega$ such that $\{t \in \Sigma T : \lambda(t) \leq |t|\} \in \Sigma\mathcal{U}$ there is $T' \preccurlyeq_{\mathcal{U}} T$ with $\lambda_{T'} =_{\Sigma\mathcal{U}} \lambda$.*

Proof We proceed by induction on the rank of T. The claim is clear if $T = \{0\}$.

If $\{t \in \Sigma T : \lambda(t) = 0\} \in \Sigma\mathcal{U}$, then $T' = \{0\}$ has the required properties. From now on we assume that $\{t \in \Sigma T : \lambda(t) > 0\} \in \Sigma\mathcal{U}$.

By RST, Definition 6.2, $X_0 := \{i : (X)_{\langle i\rangle} \in \Sigma\mathcal{U}_{\langle i\rangle}\} \in \mathcal{U}(0)$. For each $i \in X_0$, $s \in (X)_{\langle i\rangle}$ implies $\langle i\rangle \frown s \in X$, so $\lambda(\langle i\rangle \frown s) > 0$; let $\lambda_i(s) = \lambda(\langle i\rangle \frown s) - 1$. By the inductive assumption, for each $i \in X_0$ there is $T'_i \preccurlyeq_{\mathcal{U}_{\langle i\rangle}} T_{\langle i\rangle}$ and a set $X_i \subseteq (X)_{\langle i\rangle}$, $X_i \in \Sigma\mathcal{U}_{\langle i\rangle}$, such that $\lambda_{T'_i}(s) = \lambda_i(s)$ for all $s \in X_i$.

Now let $T' := \{0\} \cup \bigcup_{i\in X_0}\langle i\rangle \frown T'_i$; clearly $T' \preccurlyeq_{\mathcal{U}} T$. The set $Y := \{\langle i\rangle \frown s : i \in X_0, s \in X_i\} \in \Sigma\mathcal{U}$ and for $t = \langle i\rangle \frown s \in Y$, $\lambda_{T'}(t) = |\pi_{T',T}(t)| = |\pi_{T'_i,T_{\langle i\rangle}}(t)| + 1 = \lambda_{T'_i}(s) + 1 = \lambda_i(s) + 1 = \lambda(t)$. □

The next proposition generalizes RST, Proposition 10.5 from pedigrees to level sets.

Proposition 6.2 *Let $L = \{\gamma_0, \ldots, \gamma_n\}$ be a level set. For every x either $x \sqsubset \gamma_0$ or $\gamma_i \sqsubseteq x \sqsubset \gamma_{i+1}$ for some $i < n$, or $\gamma_n \sqsubseteq x$.*

Proof Either $x \boxminus \gamma_i$ for some $i \le n$, or $L \cup \{x\}$ is a level set, hence well-ordered by $\sqsubseteq$. From this, the claim follows. □

The following proposition is often useful for specifying subsets of level sets.

Definition 6.3 A formula $\mathcal{P}(z, \bar{x})$ is **stable in** z if $\alpha \sqsubset z \to (\mathcal{P}(z, \bar{x}) \leftrightarrow \mathcal{P}^{\alpha}(z, \bar{x}))$.

Examples. (1) $\mathcal{P}(z, X) :\ z \sqsubset x$ is stable in z. [For $\alpha \sqsubset z$, $z \sqsubset x \leftrightarrow z \sqsubset_{\alpha} x$.]

(2) Similarly, $\mathcal{P}(z, X) :\ (\exists x \in X)(x \boxminus z)$ is stable in z.

(3) $\mathcal{P}(z, X) :\ (\forall v)[z \sqsubset v \to (\exists x \in X)(z \sqsubset x \sqsubseteq v)]$ is also stable in z.

Proposition 6.4 *Let $\mathcal{P}(z, \bar{x})$ be stable in z. For every $\bar{x}$ and every level set L there is a (level) set M such that $(\forall z)(z \in M \leftrightarrow z \in L \wedge \mathcal{P}(z, \bar{x}))$.*

Proof Let $L = \{\gamma_0, \ldots, \gamma_n\}$; we consider the statement

$$\mathcal{Q}^{\alpha}(L, \bar{x}) : \quad (\exists N)(\forall z)(z \in N \leftrightarrow z \in L \wedge z \sqsupset_{\alpha} 0 \wedge \mathcal{P}^{\alpha}(z, \bar{x}))$$

and use Granularity to prove $\mathcal{Q}^{0}(L, \bar{x})$.

The statement is true when $L \sqsubseteq \alpha$, with $N = \varnothing$. [Recall that $L \sqsubseteq \alpha$ implies $z \sqsubseteq \alpha$ for all $z \in L$; see RST2, Proposition 1.10 (13); also note that $L \boxminus \gamma_n$.]

By Granularity, there is a $\sqsubseteq$-least level α for which $\mathcal{Q}^\alpha(L,\overline{x})$ holds; by the above, $\alpha \sqsubseteq \gamma_n \boxminus L$.

By Proposition 6.2, either $\alpha \sqsubset \gamma_0$ or $\gamma_i \sqsubseteq \alpha \sqsubset \gamma_{i+1}$ for some $i < n$, or $\alpha \boxminus \gamma_n$.

In the first case, $\mathcal{Q}^\alpha(L,\overline{x})$ clearly implies $\mathcal{Q}^0(L,\overline{x})$, with the same N. In the second case, $\mathcal{Q}^\alpha(L,\overline{x})$ implies $\mathcal{Q}^{\gamma_i}(L,\overline{x})$ (with the same N), so $\alpha \boxminus \gamma_i$. Let $\gamma_{i-1} \sqsubseteq \beta \sqsubset \gamma_i$ [$\beta \sqsubset \gamma_i$ if $i = 0$ and $\gamma_i \sqsupset 0$]. By stability of $\mathcal{P}$ in z, $\mathcal{P}^\beta(z,\overline{x}) \leftrightarrow \mathcal{P}(z,\overline{x}) \leftrightarrow \mathcal{P}^\alpha(z,\overline{x})$ for all $z \sqsupset \alpha$. If $\mathcal{P}^\beta(\gamma_i,\overline{x})$, let $N' := N \cup \{\gamma_i\}$; otherwise let $N' := N$. Then $\mathcal{Q}^\beta(L,\overline{x})$ holds (with N' in place of N), and we have a contradiction. Thus $\alpha \boxminus \gamma_0 \boxminus 0$, and $\mathcal{Q}^0(L,\overline{x})$ holds in this case, too. The third case is like the second, with $i = n$.

Let now N be such that $(\forall z)(z \in N \leftrightarrow z \in L \wedge z \sqsupset 0 \wedge \mathcal{P}(z,\overline{x}))$.

We set $M := N \cup \{\gamma_0\}$ if $\gamma_0 \boxminus 0 \wedge \mathcal{P}(\gamma_0,\overline{x})$, and $M := N$ otherwise. Clearly M has the required properties. □

In RST, Corollary 12.7, the completeness of **GRIST** over **ZFC** is formulated as follows:

If $\mathbf{T} \supseteq$ **ZFC** *is a complete consistent theory (in the* $\in$*-language), then* $\mathbf{T} +$ **GRIST** *is a complete consistent theory (in the* $\in$-$\sqsubseteq$*-language).*

Here we give a reformulation that is perhaps more striking.

Theorem 6.5 *Let* $\mathcal{P}$ *be any formula (in the* $\in$-$\sqsubseteq$*-language). If* **GRIST** $+ \mathcal{P}$ *is a conservative extension of* **ZFC**, *then* **GRIST** $\vdash \mathcal{P}$.

Proof Assume that **GRIST** $+ (\neg\mathcal{P})$ is consistent. Let $\overline{\mathbf{G}}$ be a complete consistent extension of this theory, and let $\overline{\mathbf{T}}$ be the restriction of $\overline{\mathbf{G}}$ to formulas in the $\in$-language. Then $\overline{\mathbf{T}} \supset$ **ZFC** is complete and consistent. By RST, Corollary 12.7, $\overline{\mathbf{T}} +$ **GRIST** is complete and consistent, so $\overline{\mathbf{T}} +$ **GRIST** $= \overline{\mathbf{G}}$ and $\overline{\mathbf{T}} +$ **GRIST** $\vdash \neg\mathcal{P}$. Hence $\overline{\mathbf{T}} +$ **GRIST** $+ \mathcal{P}$ is inconsistent. It follows that **GRIST**$+ \mathcal{P} \vdash \neg\mathcal{Q}$ for some $\mathcal{Q} \in \overline{\mathbf{T}}$. Since **GRIST**$+ \mathcal{P}$ is assumed to be a conservative extension of **ZFC**, we have also **ZFC** $\vdash \neg\mathcal{Q}$ and $\neg\mathcal{Q} \in \overline{\mathbf{T}}$, a contradiction. □

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Received: 25 April 2011 Revised: 16 May 2012